MATH 240 Qualifying Exam
September 14, 2021

Instructions: 3 hours, open book/notes (only Folland or personal lecture notes; no HW or other solutions). You may use without proofs results proved in Folland up to Section 8.3. Present your solutions clearly, with appropriate detail.

1. (30 pts) Let \( f \in C(\mathbb{R}) \) and let \( A \subseteq \mathbb{R} \) be a Borel set such that \( f \) is differentiable at each \( x \in \mathbb{R} \setminus A \) and \( f'(x) = 0 \) for all such \( x \).

(a) If \( A \) is closed and countable, show that \( f \) is constant.

(b) If \( A \) has Lebesgue measure 0, must \( f \) be constant? Prove or find a counterexample.

2. (30 pts) Does there exist a Borel measurable function \( f : \mathbb{R} \to [0, \infty) \) such that \( \int_a^b f(x) \, dx = \infty \) for all real numbers \( a < b \)? Either find an example or show that no such \( f \) exists.

3. (30 pts) Let \( p \in (1, \infty) \), and for \( f \in L^p(\mathbb{R}) \) define \( T f(x) := \int_0^1 f(x + y) \, dy \).

(a) Show that \( \|T f\|_p \leq \|f\|_p \), and equality holds if and only if \( f = 0 \) almost everywhere.

(b) Prove that \( (I - T)(L^p(\mathbb{R})) \neq L^p(\mathbb{R}) \), where \( I \) is the identity map on \( L^p(\mathbb{R}) \).

4. (30 pts) Let functions \( f_n \in C([0, 1]) \) satisfy \( \sup_n |f_n(x)| < \infty \) for each \( x \in [0, 1] \). Show that there are \( 0 \leq a < b \leq 1 \) such that \( \sup_n \|f_n \chi_{(a,b)}\|_u < \infty \).

5. (20 pts) Let \( f_n, f \in L^2(\mathbb{R}) \) satisfy \( f_n \rightarrow f \) weakly and \( \|f_n\|_2 \rightarrow \|f\|_2 \) as \( n \rightarrow \infty \). Show that \( f_n \rightarrow f \) in \( L^2(\mathbb{R}) \).

6. (30 pts) Let \( \delta_x \) denote the Dirac delta mass at \( x \in \mathbb{R}^n \). Let \( \{x_j\}_{j=1}^\infty \) be a sequence in \( \mathbb{R}^n \), \( \{c_j\}_{j=1}^\infty \) a sequence of positive numbers, and \( \mu \) the Borel measure on \( \mathbb{R}^n \) corresponding to the series \( \sum_{j=1}^\infty c_j \delta_{x_j} \). Prove that \( \mu \) is Radon if and only if for all convergent subsequences \( \{x_{j_k}\}_{k=1}^\infty \) it holds that \( \sum_{k=1}^\infty c_{j_k} < \infty \).

7. (30 pts) For any \( f \in L^2(\mathbb{R}) \cap C^1(\mathbb{R}) \) show that
\[
\left( \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right) \left( \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi \right) \geq \frac{1}{16\pi^2} \left( \int_{\mathbb{R}} |f(x)|^2 \, dx \right)^2.
\]
Here \( \hat{f} \) is the Fourier transform of \( f \) and \( dx, d\xi \) represent the Lebesgue measure on \( \mathbb{R} \).