Complex Analysis Qualifying Exam – Fall 2021

Name: ________________________________

Student ID: __________________________

Instructions:

You do not have to reprove any results from Conway. However, if using a homework problem, please make sure you reprove it.

You have 180 minutes to complete the test.

Notation: $\Delta = \{ z \in \mathbb{C} \mid |z| < 1 \}$.

<table>
<thead>
<tr>
<th>Question</th>
<th>Score</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>70</td>
</tr>
</tbody>
</table>
**Problem 1.** [10 points.]

Compute the following integral via residues

\[ \int_{0}^{\infty} \frac{1 - \cos x}{x^2} \, dx. \]

Please explain the necessary estimates.
Problem 2. [10 points; 5, 5.]

Let \( K = \{ z \in \mathbb{C} : |z| \leq 3, |z - 1| \geq 1, |z + 1| \geq 1 \} \).

(i) True/false: every holomorphic function in a neighborhood of \( K \) is the local uniform limit on \( K \) of a sequence of polynomials. Please justify your answer.

(ii) Determine, with justification, the set

\[
\hat{K} = \{ z \in \mathbb{C} : |p(z)| \leq \sup_{w \in K} |p(w)| \text{ for all polynomials } p \}.
\]
Problem 3. [10 points; 5, 5.]

Let $a, b : \mathbb{C} \to \mathbb{C}$ be entire functions. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that

$$f(z)^2 + a(z)f(z) + b(z) = 0.$$ 

(i) Show that if $a, b$ have finite order, then $f$ is also of finite order.

(ii) Show that if $a, b$ are polynomials, then $f$ is also a polynomial.
**Problem 4.** [10 points.]

Let $a_n \in \Delta$ be a sequence such that $a_n \to 1$. Let $f_n : \Delta \to \Delta$ be a sequence of holomorphic functions such that $f_n(0) = a_n$. Show that $f_n \to 1$ uniformly on compact subsets of $\Delta$. 
Problem 5. [10 points; 5, 5.]

Let $G = \{ z = x + iy : x > 0, y > 0, xy < 1 \}$.

(i) Construct a biholomorphism between $G$ and the trip $S = \{ z = x + iy, 0 < y < 1 \}$.

(ii) Construct an unbounded continuous function $u : \overline{G} \to \mathbb{R}$, harmonic in $G$, and such that $u$ vanishes on $\partial G$. 
Problem 6. [10 points.]

Let \( f : \mathbb{C} \to \mathbb{C} \) be an entire function such that \( |f(z)| = 1 \) for \( |z| = 1 \). Show that there exists \( a \in \mathbb{C} \) and \( n \geq 0 \) such that \( f(z) = az^n \).
Problem 7. [10 points.]

Describe all entire functions \( f : \mathbb{C} \to \mathbb{C} \) such that for all \( z \in \mathbb{R} \) we have \( |f(z)| = 1 \).