1. (25 pts) Let $A, B \subseteq \mathbb{R}$ be Lebesgue measurable with $\{a + b | a \in A, b \in B\} \subseteq \mathbb{R} \setminus \mathbb{Q}$. Prove that either $m(A) = 0$ or $m(B) = 0$.

2. (40 pts) Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and let $F \subseteq L^1(\mu)$. We say that $F$ is uniformly integrable if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\left| \int_E f \, d\mu \right| < \varepsilon$ whenever $f \in F$ and $E \in \mathcal{M}$ satisfies $\mu(E) < \delta$.

   (a) If $p \in (1, \infty]$ and $F \subseteq L^p(\mu)$ is bounded, prove that $F$ is uniformly integrable.

   (b) Give an example where (a) fails for $p = 1$.

   (c) Let $f_n, f \in L^1(\mu)$ and assume that $F := \{f_1, f_2, \ldots\}$ is uniformly integrable. If $f_n \to f$ in measure, prove that $f_n \to f$ in $L^1(\mu)$.

3. (25 pts) If $f \in L^2([0, 1])$ and $\int_0^1 x^{2n} f(x) \, dx = \frac{1}{2n+2}$ for $n = 0, 1, \ldots$, must $f(x) = x$ a.e.?

4. (25 pts) Let $X, Y, Z$ be Banach spaces and $B : X \times Y \to Z$ be a map such that for any fixed $x \in X$ we have $B(x, \cdot) \in L(Y, Z)$ and for any fixed $y \in Y$ we have $B(\cdot, y) \in L(X, Z)$. Show that there is $C \geq 0$ such that $\|B(x, y)\| \leq C\|x\|\|y\|$ for all $(x, y) \in X \times Y$.

5. (25 pts) Find all $f \in L^2([-1, 1])$ such that

   $$\int_{-1}^1 |f(x) - \sqrt{3} x|^2 \, dx \leq \frac{1}{4}$$

   and

   $$\int_{-1}^1 |f(x) - \sqrt{5} x^2|^2 \, dx \leq \frac{9}{4}.$$  

6. (30 pts) Let $X$ be an LCH space, and $Y$ a closed subset of $X$. Show that if $\mu$ is a Radon measure on $Y$, then $\nu(E) := \mu(E \cap Y)$ defines a Radon measure $\nu$ on $X$. Also demonstrate that $Y$ being closed is needed here, by giving an example where $Y$ is not closed and the corresponding $\nu$ is not Radon.

7. (30 pts) Let $\mu$ be a $\sigma$-finite Radon measure on an LCH space $X$, and $\varphi$ a positive continuous function on $X$. Show that $\nu(E) := \int_E \varphi \, d\mu$ defines a Radon measure $\nu$ on $X$.

   **Hint:** First consider the positive linear functional $I(f) := \int f \varphi \, d\mu$ on $C_c(X)$ and show that $\nu$ coincides with the Radon measure associated with this functional on open sets.