MATH 240 Qualifying Exam

May 10, 2021

Instructions: 3 hours, open book/notes (only Folland or personal lecture notes; no HW or other solutions). You may use without proofs results proved in Folland Chapters 1-7. Present your solutions clearly, with appropriate detail.

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1. (25 pts) Let $A, B \subseteq \mathbb{R}$ be Lebesgue measurable with $\{a+b \,|\, a \in A,\, b \in B\} \subseteq \mathbb{R} \setminus \mathbb{Q}$. Prove that either $m(A) = 0$ or $m(B) = 0$.

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2. (40 pts) Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and let $\mathcal{F} \subseteq L^1(\mu)$. We say that $\mathcal{F}$ is *uniformly integrable* iff for every $\varepsilon > 0$ there is $\delta > 0$ such that $|\int_E f d\mu| < \varepsilon$ whenever $f \in \mathcal{F}$ and $E \in \mathcal{M}$ satisfies $\mu(E) < \delta$.

(a) If $p \in (1, \infty]$ and $\mathcal{F} \subseteq L^p(\mu)$ is bounded, prove that $\mathcal{F}$ is uniformly integrable.

(b) Give an example where (a) fails for $p = 1$.

(c) Let $f_n, f \in L^1(\mu)$ and assume that $\mathcal{F} := \{f_1, f_2, \dots\}$ is uniformly integrable. If $f_n \to f$ in measure, prove that $f_n \to f$ in $L^1(\mu)$.

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3. (25 pts) If $f \in L^2([0,1])$ and $\int_0^1 x^{2n} f(x) dx = \frac{1}{2n+2}$ for $n = 0, 1, \dots$, must $f(x) = x$ a.e.?

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4. (25 pts) Let X, Y, Z be Banach spaces and $B : X \times Y \to Z$ be a map such that for any fixed $x \in X$ we have $B(x, \cdot) \in L(Y, Z)$ and for any fixed $y \in Y$ we have $B(\cdot, y) \in L(X, Z)$. Show that there is $C \geq 0$ such that $\|B(x,y)\| \leq C\|x\|\,\|y\|$ for all $(x, y) \in X \times Y$.

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5. (25 pts) Find all $f \in L^2([-1,1])$ such that

$$\int_{-1}^{1} \left| f(x) - \sqrt{3}\, x \right|^2 dx \leq \frac{1}{4}$$

and

$$\int_{-1}^{1} \left| f(x) - \sqrt{5}\, x^2 \right|^2 dx \leq \frac{9}{4}.$$

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6. (30 pts) Let X be an LCH space, and Y a closed subset of X. Show that if μ is a Radon measure on Y, then $\nu(E) := \mu(E \cap Y)$ defines a Radon measure ν on X. Also demonstrate that Y being closed is needed here, by giving an example where Y is not closed and the corresponding ν is not Radon.

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7. (30 pts) Let μ be a σ-finite Radon measure on an LCH space X, and φ a positive continuous function on X. Show that $\nu(E) := \int_E \varphi \, d\mu$ defines a Radon measure ν on X.

Hint: First consider the positive linear functional $I(f) := \int f\varphi \, d\mu$ on $C_c(X)$ and show that ν coincides with the Radon measure associated with this functional on open sets.
