Complex Analysis Qualifying Exam – Spring 2021

Name: ____________________________________________

Student ID: _______________________________________

**Instructions:**

The exam is closed notes, closed books, no internet, no outside help.

You do not have to reprove any results from Conway or shown in class. However, if using a homework problem, please make sure you reprove it.

You have 180 minutes to complete the test.

**Notation:** $\Delta = \{ z \in \mathbb{C} \mid |z| < 1 \}$.

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Problem 1. [10 points.]

How many solutions, counted with multiplicities, does the equation

\[ z^3 \sin z + 5z^2 + 2 = 0 \]

have in the unit disc \(|z| < 1|\)?
Problem 2. [10 points; 5, 5.]

Let $K$ be a proper closed arc of the unit circle $|z| = 1$.

(i) Is there a sequence of polynomials $P_n(z)$ such that $P_n(z) \to \bar{z}$ uniformly in $K$?

(ii) Is there a sequence of polynomials $P_n(z)$ such that $P_n(z) \to \bar{z}$ uniformly on the circle $|z| = 1$?

Please justify your answers.
Problem 3. [10 points.]

Let $\mathcal{F}$ denote the family of holomorphic functions $f : \Delta \to \mathbb{C}$ such that

(i) $f$ omits all strictly negative real numbers, and

(ii) $f(0) = 1$.

Find the maximum value of $|f'(0)|$ as $f \in \mathcal{F}$. 
Problem 4. [10 points.]

Let \( f : G \to \mathbb{C} \) be a holomorphic function in \( G = \{ z : |z| < 2 \} \) such that \( |f(z)| < 1 \) for \( z \in G \).
Assume that
\[
 f(1) = f(-1) = f(i) = f(-i) = 0.
\]
Show that
\[
 |f(0)| \leq \frac{1}{15}.
\]
Problem 5. [10 points.]

Let $a_n = 1 - \frac{1}{n}$ for $n \geq 2$. Show that there are no bounded holomorphic functions $f : \Delta \to \mathbb{C}$ with zeros only at the $a_n$'s.
Problem 6. [10 points.]

Let \( \{u_n(x, y)\} \) be a sequence of harmonic functions in an open connected set \( G \subset \mathbb{R}^2 \), converging uniformly on compact subsets of \( G \).

Show that the sequence of partial derivatives \( \frac{\partial u_n}{\partial x} \) converges uniformly on compact subsets of \( G \).
Problem 7. [10 points.]

Let \( \{ f_n \} \) be a sequence of automorphisms of the unit disc \( \Delta \), converging locally uniformly in \( \Delta \) to a nonconstant function \( f \). Show that \( f \) is an automorphism of \( \Delta \).

*Hint:* Examine the family \( \mathcal{F} \) consisting of the inverse automorphisms \( f_n^{-1} : \Delta \to \Delta \).