Applied Algebra Qualifying Exam: Part A

5:00pm–8:00pm (PDT), via Zoom. Meeting ID: 912 8480 3260
Tuesday May 11th, 2021

• Write your name and student PID at the top right corner of each page of your submission.

• Do all four problems. Show your work.

• This part of the exam will represent 40% of the total score.

• Your completed examination must be uploaded to Gradescope while you are connected to Zoom. You may leave the meeting once the Proctor has checked that your exam has been uploaded.

• It is your responsibility to check that any uploaded material is both complete and legible.

• By participating in this exam you are agreeing to abide by the UCSD Policy on Academic Integrity. The instructors reserve the right to require a follow-up oral examination.

• This is a closed-book examination. No cell-phone or Internet aids.

• Please keep your camera turned on throughout the exam.

• Notation:
  – $\mathcal{M}_{m,n}$ denotes the set of $m \times n$ matrices with complex components.
  – $\mathcal{M}_n$ denotes the set $\mathcal{M}_{m,n}$ with $m = n$.
  – $\mathbb{C}^n$ is the set of column vectors with $n$ complex components.
  – $x^H$ is the Hermitian transpose of a vector or matrix $x$.
  – $\text{eig}(A)$ is the set of eigenvalues of the matrix $A$ (counting multiplicities).
Question 1.

(a) (4 points) State, but do not prove, the Schur decomposition theorem for a matrix $A \in M_n$.

(b) (8 points) Let $(\lambda, x)$ be a simple eigenpair of $A \in M_n$ with $x^H x = 1$. Prove that there exists a nonsingular matrix $(x\ X)$ with inverse $(y\ Y)^H$ such that

$$
\begin{pmatrix}
y^H \\
y^H
\end{pmatrix}
A
\begin{pmatrix}
x & X
\end{pmatrix}
=
\begin{pmatrix}
\lambda & 0 \\
0 & M
\end{pmatrix}.
$$

(c) (8 points) Hence prove that the angle $\theta$ between $x$ and $y$ satisfies $\sec \theta = \|y\|_2$. 

Question 2.

(a) (8 points.) Consider any Hermitian $A \in \mathcal{M}_n$ with eigenvalues ordered so that $\lambda_n(A) \leq \cdots \leq \lambda_2(A) \leq \lambda_1(A)$. Prove that

$$\lambda_n = \min_{x \neq 0} \frac{x^H A x}{x^H x}.$$ 

(b) (12 points) Suppose that $D \in \mathcal{M}_n$ with $D = \text{diag}(d_1, d_2, \ldots, d_n)$. Prove that for all $1 \leq p \leq \infty$ the $p$-norm of $D$ is given by $\|D\|_p = \max_{1 \leq i \leq n} |d_i|$. 
Question 3.

(a) (4 points.) State, but do not prove, the singular-value decomposition theorem.

(b) (8 points.) For a given $A \in \mathcal{M}_{m,n}$, prove that

$$\sigma_1(A) = \max_{x,y \neq 0} \frac{|y^H A x|}{\|y\|_2 \|x\|_2},$$

where $\sigma_1(A)$ is the largest singular value of $A$.

(c) (8 points.) For any $A \in \mathcal{M}_n$, define (i) the field of values $\mathcal{F}(A)$; (ii) the spectral radius $\rho(A)$; and the numerical radius $\omega(A)$. Prove that $\rho(A) \leq \omega(A) \leq \sigma_1(A)$. 

Question 4.

(a) (8 points) Let $C \in \mathcal{M}_{m,n}$ with rank($C$) = $m$. Find orthogonal projections that project $x \in \mathbb{C}^n$ onto range($C^H$) and null($C$). Verify that your projections satisfy the properties of an orthogonal projection.

(b) For a given nonzero $y \in \mathbb{C}^n$, let $\mathcal{Y} = \text{span}(y)$.
   
   (i) (6 points) Find an *oblique* projector $A$ that project vectors onto $\mathcal{Y}$. Find the complementary projection.

   (ii) (6 points) Find the unique orthogonal projector $A$ that projects vectors onto $\mathcal{Y}$. Find the complementary projection associated with $A$. 
Instructions: Do all problems. All problems are weighted equally. You are not allowed to consult any external resource during this exam. Good luck!

Problem 1: Let $G$ be a group (possibly infinite) and let $V$ be a finite-dimensional $G$-module over $\mathbb{C}$. Assume that $V$ admits a $G$-invariant inner product $\langle -, - \rangle$. Prove that $V$ is completely reducible.
Problem 2: Let $\mathbb{R}_+$ be the group of positive real numbers under multiplication. Is every indecomposable $\mathbb{R}_+$-module over the complex numbers irreducible?
Problem 3: Let $\lambda, \mu \vdash n$ be partitions and let $S^\lambda, S^\mu$ be the corresponding irreducible $S_n$-modules. Endow the tensor product $S^\lambda \otimes S^\mu$ with the structure of an $S_n$-module by the rule

$$\sigma \cdot (v \otimes w) := (\sigma \cdot v) \otimes (\sigma \cdot w)$$

for $\sigma \in S_n, v \in S^\lambda, w \in S^\mu$. Find the vector space dimension of the $S_n$-fixed subspace

$$(S^\lambda \otimes S^\mu)^{S_n}$$

of $S^\lambda \otimes S^\mu$. 
\textbf{Problem 4}: Find the character table of the alternating subgroup $A_4$ of the symmetric group $S_4$. The group algebra of $A_4$ is isomorphic to a direct sum
\[
\mathbb{C}[A_4] \cong \text{Mat}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \text{Mat}_{n_r}(\mathbb{C})
\]
of matrix algebras over $\mathbb{C}$. Determine $r$ and the numbers $n_1, \ldots, n_r > 0$. 
Applied Algebra Qualifying Exam: Part C

5:00pm–8:00pm (PDT), via Zoom. Meeting ID: 912 8480 3260
Tuesday May 11th, 2021

• Write your name and student PID at the top right corner of each page of your submission.

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Question 1.

(a) (2 points) Let $B(n)$ be the permutation group generated by the transpositions $\tau_i = (2i - 1 \ 2i)$, $1 \leq i \leq n$. Show that every character $\chi$ of $B(n)$ takes values in $\{-1, 1\}$. 
(b) (8 points) Explicitly describe the dual group of $B(n)$. 
Question 2.

(a) (2 points.) With notation as in the previous problem, give the definition of the Cayley graph of $B(n)$ as generated by $\tau_1, \ldots, \tau_n$. 
(b) (8 points) Compute the eigenvalues and eigenvectors of the adjacency operator of the graph in part (a).