Algebra fall qualifying exam, August 31, 2020.

All problems are worth 15 points.

1. Suppose that G is a finite group that has a normal subgroup N with the following properties: the order of N is n, G/N is a cyclic group of order m, and $\gcd(m, \phi(n)) = 1$ where $\phi(n) := |\{k | 1 \leq k \leq n, (k, n) = 1\}|$.

(1) Prove that if N is cyclic, then G is abelian.
(2) Give an example where N is abelian, but G is not.

2. Suppose that G is a group with $|G| = 2^k m$, where m is odd and $k \geq 0$. Assume that G has a cyclic Sylow 2-subgroup. Prove that G has a characteristic subgroup H which has order m.

3. Determine if each of the following rings is a unique factorization domain. For each case, you need give only a short justification or line of argument.

(1) $\mathbb{Z}[2\sqrt{2}]$.
(2) $\mathbb{Z}[x, y]$.
(3) $\mathbb{Z} + x\mathbb{Q}[x] := \{a_0 + a_1 x + \cdots + a_n x^n |\, a_0 \in \mathbb{Z}, a_1, \ldots, a_n \in \mathbb{Q}, n \in \mathbb{Z}^+\}$ (hint: consider the element x).

4. Let F be an algebraically closed field. Let A be the $n \times n$ matrix over F such that every entry of A is 1. Find the Jordan canonical form of A. (The answer may depend on the properties of the field F).

5. Suppose that D is an integral domain and M is a D-module.

(1) Prove that if M is flat, then it is torsion-free.
(2) Prove that if D is a PID, and M is finitely generated and torsion-free, then M is flat.

6. Let A be a commutative unital ring. Let $\{a_1, \ldots, a_n\} \subset A$ be such that the ideal generated by $\{a_1, \ldots, a_n\}$ equals A. For every $1 \leq i \leq n$, put $S_i = \{1, a_i, a_i^2, \ldots\}$. Let M be an A-module with submodule N, and assume that for all $1 \leq i \leq n$ we have $S_i^{-1}M = S_i^{-1}N$. Prove that $N = M$. (Hint: for $x \in M$ consider $\{a \in A \,|\, ax \in N\}$.)

7. Let K be the splitting field over $\mathbb{Q}$ of $f(x) = x^4 - 4x^2 + 1$.

Find $\mathrm{Gal}(K/\mathbb{Q})$, and find all fields E such that $\mathbb{Q} \subsetneq E \subsetneq K$.