QUALIFYING EXAMS

Spring 2020

Three-hour exam. Do as many questions as you can. No book or notes allowed. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly. **Even if you can not solve the whole problem, you still need to write your partial answer to receive partial credit.**

1. Consider the group $G = \mathbb{Z} * (\mathbb{Z} \oplus \mathbb{Z})$ (here $*$ means the free product). Realize G as the fundamental group of some topological space. Then classify (up to isomorphism) all subgroups of G of index two.

2. Let n be a positive even number. Show that there does not exist a continuous map $f : S^n \to S^n$ such that for any $\vec{x} \in S^n$, we have $f(\vec{x}) \neq \vec{x}$ and $f(\vec{x}) \neq -\vec{x}$. (Here $-\vec{x}$ denotes the antipodal point of $\vec{x}$.)

3. Let X be a 3-dimensional CW complex obtained by attaching a single 3-dimensional cell to S^2 via an attaching map of degree 3. Compute the homology group $H_k(X \times X; \mathbb{Z})$ for all $k \geqslant 0$.

4. Show that $\mathbb{C}P^2$ has no orientation reversing self-homeomorphism.

5. Let M be a closed, orientable, connected manifold of dimension n. Suppose there is a continuous map $f : S^n \to M$ with nonzero mapping degree. Show that $H_k(M; \mathbb{Q}) = H_k(S^n; \mathbb{Q})$ for any $k \geqslant 0$. (Hint: use the Poincaré duality.)

6. Consider a continuous map $f : \mathbb{R}P^n \to \mathbb{R}P^n$, where n is a positive even number. Show that f has a fixed point. (Here $\mathbb{R}P^n$ is the n-dimensional real projective space obtained by identifying antipodal points of the sphere S^n.)

7. Compute the homotopy group $\pi_3(\mathbb{T}^2 \vee S^3)$. Here $\vee$ means one-point union of two topological spaces (wedge) and $\mathbb{T}^2$ denotes the 2 dimensional torus.

8. Let X be an n-dimensional closed smooth manifold. Suppose X is simply connected and suppose

$$H_k(X; \mathbb{Z}) = H_k(S^n; \mathbb{Z})$$

for all $k \geqslant 0$. Show that X is homotopy equivalent to S^n.