ALGEBRA QUALIFYING EXAM, SPRING 2020

All problems are worth 15 points.

1. (a) Prove that $\langle a, b | a^2, b^2 \rangle$ is isomorphic to the group of Euclidean symmetries of $\mathbb{Z}$. (You can use without proof that the group of Euclidean symmetries of $\mathbb{Z}$ is

$$\{f : \mathbb{Z} \to \mathbb{Z} |\ f(x) = ax + b, a = \pm 1, b \in \mathbb{Z}\}$$

and it is generated by the reflection $f_0(x) := -x$ and the translation $g_0(x) := x + 1$.)

(b) Prove that any group generated by two elements of order 2 is solvable.

2. Suppose that G is a finite group and P is a p-subgroup of G for some prime p. Prove that

$$\left|\{Q \in \mathrm{Syl}_p(G) |\ P \subseteq Q\}\right| \equiv 1 \pmod{p},$$

where $\mathrm{Syl}_p(G)$ is the set of Sylow p-subgroups of G.

3. Let R be a noetherian integral domain. Show that the following conditions are equivalent:

(1) Every finitely generated R-module is a direct sum of cyclic R-modules.
(2) R is a PID.

4. Suppose that A is a unital commutative ring without any non-zero nilpotent elements. Let $N \in \mathrm{Mat}_n(A)$ be a nilpotent element. Prove that $N^n = 0$. (Hint: Prove that $N^n \in \mathrm{Mat}_n(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ in A).

5. Suppose A is a unital commutative ring.

(a) Let M and N be two submodules of an A-module K. Suppose $M + N$ and $M \cap N$ are finitely generated A-modules. Prove that M is a finitely generated A-module.

(b) Let $\Sigma := \{\mathfrak{a} \trianglelefteq A |\ \mathfrak{a}$ is not a finitely generated ideal$\}$. Suppose Σ is not empty. Prove that Σ has a maximal element.

(c) Let $\mathfrak{p}$ be a maximal element of Σ. Prove that $\mathfrak{p}$ is a prime ideal. (Hint: Suppose to the contrary that $ab \in \mathfrak{p}$ and $a, b \notin \mathfrak{p}$ for some $a, b \in A$; consider $\mathfrak{p} + \langle a \rangle$ and $\mathfrak{p} \cap \langle a \rangle$.)

6. Let F be a field with algebraic closure $\overline{F}$. Let $F \subseteq K \subseteq \overline{F}$ and $F \subseteq L \subseteq \overline{F}$, where K and L are fields with $[K : F] < \infty$ and $[L : F] < \infty$. Prove that the following conditions are equivalent:

(1) $K \otimes_F L$ is a field.

(2) Given any F-linearly independent elements $\alpha_1, \dots \alpha_m \in K$, then $\alpha_1, \dots, \alpha_m$ are linearly independent over L.

7. Let p be a fixed prime. Suppose that F is a field with the following property: given any field extension $F \subseteq K$ with $[K : F] < \infty$, then $[K : F]$ is divisible by p.

(a) Suppose that $F \subseteq K$ is a separable field extension with $[K : F] < \infty$. Show that $[K : F]$ is a power of p.

(b) Show that either F is a perfect field or else $\mathrm{Char}(F) = p$.