QUALIFYING EXAMS

Fall 2019

Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly.

1. Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a continuous map. Suppose that there exists a uniform constant C such that $\|f(\vec{x}) - \vec{x}\| \leqslant C$ for any $\vec{x} \in \mathbb{R}^n$. (Here $\|\cdot\|$ denotes the standard norm of a vector.) Prove that f is surjective. (It would be helpful to start with the simple case $n = 2$.)

2. Let G_n be the group with generators $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots b_n\}$ and a single relation

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} = 1.$$

Classify (up to isomorphism) all subgroups of G_n with index m. (Hint: Use the fact that each subgroup of the fundamental group corresponds to a connected covering space.)

3. Compute the integer homology $H_*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z})$.

4. Recall definition of the complex projectile space $\mathbb{C}P^n = (\mathbb{C}^{n+1} - \{\vec{0}\})/\sim$, where the equivalence relation $\sim$ is defined as follows:

- $(z_0, \cdots, z_n) \sim (w_0, \cdots, w_n)$ if and only if there exists a nonzero complex number a such that $z_i = a \cdot w_i$ for all $0 \leqslant i \leqslant n$.

When n is even, show that any continuous map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point.

5. Let M be a connected, closed n-dimensional manifold. Show that $H_{n-1}(M; \mathbb{Z})$ is torsion-free if and only if M is orientable.

6. Let M, N be two closed, oriented, n-dimensional manifolds. Show that the mapping degree of a continuous map $f : M \to N$ must be divisible by $[\pi_1(N), f_*(\pi_1(M))]$, the index of the subgroup $f_*(\pi_1(M))$ in $\pi_1(N)$. (Recall that f has degree k if it sends the fundamental class $[M] \in H_n(M; \mathbb{Z})$ to k-times of the fundamental class $[N]$.)

7. Let f ang g be two continuous maps from the three-torus $T^3 = S^1 \times S^1 \times S^1$ to the three-sphere S^3. Show that f is homotopic to g if and only if they have the same mapping degree.

8. Let M be a compact, orientable 3-dimensional manifold. Suppose the boundary of M is a surface Σ of genus g. Let $i_* : H_1(\Sigma; \mathbb{Q}) \to H_1(M; \mathbb{Q})$ be the map induced by the inclusion of the boundary. Show that the dimension of $\ker i_*$ equals g. (This is the so-called "half die half alive lemma".)