MATH 281BC – Qualifying Exam – Spring 2019

Use the notation defined in the reference sheet as much as possible; otherwise, define any symbol you use. Name any result you use from the reference sheet. Be concise and clear. All tests are considered at some prescribed level α unless otherwise specified.

Problem 1. Consider a setting where $\theta \sim \mathcal{N}(0, \tau^2)$ and, given θ, $X_1, \dots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\theta, \sigma^2)$, with τ^2 and σ^2 both known (for now). We consider the problem of estimating θ under square error loss.

A. Derive the Bayes estimator.

B. Derive the Bayes risk.

C. Prove that the sample mean (denoted $\bar{X}$) is minimax.

D. Prove that $\bar{X}$ is admissible.

E. Is $\bar{X}$ still minimax when in addition we know that $\theta \geq 0$?

Problem 2. Consider $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(\theta, \sigma^2)$ where σ^2 is known. Our goal is to estimate θ with the absolute loss function $L(\theta, d) = |\theta - d|$.

A. Derive an estimator that has minimum risk among unbiased estimators.

B. Exhibit a nontrivial group G of transformations that leave the estimation problem invariant. Explain why this group leaves the problem invariant, and define $\bar{G}$ and G^*.

C. Derive an estimator with minimum risk among equivariant estimators.

Problem 3. Suppose that $X = (X_1, \ldots, X_d)$ is multivariate normal in dimension d with mean vector $\theta = (\theta_1, \ldots, \theta_d)$ and covariance matrix $\mathbf{C} = (c_{ij})$. We are interested in testing $a^\top\theta \le q$ versus $a^\top\theta > q$, where $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$ and $q \in \mathbb{R}$ are given. We assume that $\mathbf{C}$ is known.

A. First assume that $\mathbf{C}$ is the identity matrix and that $a = (1, 0, \ldots, 0)$. Derive a uniformly most powerful test.

B. Reduce to the special case of Part A to derive a uniformly most powerful test in the general case where **C** is still the identity matrix while now a is an arbitrary (but known) nonzero vector.

C. Reduce to the special case of Part A (or B) to derive a uniformly most powerful test in the general case where **C** is an arbitrary (but known) positive definite matrix and a is an arbitrary (but known) nonzero vector.

D. What happens when **C** is singular?