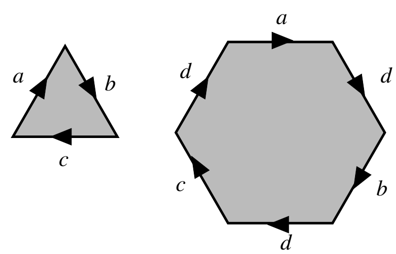
## 32. Fall 2018

*Three-hour exam. Do as many questions as you can. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly.*

**1.** Let $G$ be a topological group and let 1 be its identity element. Show that the fundamental group $\pi_1(G, 1)$ is abelian.

**2.** Let $X$ be the space made by gluing the edges of a solid hexagon $H$ and a solid triangle $T$ according to the scheme pictured below. Calculate $\pi_1 X)$ and $H_*(X; \mathbb{Z})$.




**3.** Let $Y$ be a space obtained by attaching a 4-ball $B^4$ to a 3-sphere via a degree 6 map $\phi : \partial B^4 \to S^3$. Calculate the integral homology $H_*(Y \times \mathbb{R}P^2; \mathbb{Z})$.

**4.** Provided the sum is finite, the Euler characteristic $\chi(X)$ of a space $X$ is defined to be the alternating sum of the dimensions of the rational homology groups $H_i(X; \mathbb{Q})$. Use Poincaré duality to show that the Euler characteristic of a compact connected closed orientable 3-manifold $M^3$ is zero. Prove that the result still holds even if $M$ is non-orientable. (You can assume that a compact manifold is homotopy-equivalent to a finite CW complex.)

**5.** Show that any homotopy equivalence from $\mathbb{C}P^{2n}$ to itself is orientation-preserving, that is has degree $+1$.

**6.** Calculate the homotopy group $\pi_3(\mathbb{R}P^4 \vee S^3)$. (Here $\vee$ denotes the one-point union of the two spaces).

**7.** Let $M^3$ be a *homology sphere*: a connected closed compact 3-manifold with the same homology groups as $S^3$. Calculate the fundamental group and homology of the suspension $\Sigma M$? Use this to show that the suspension is homotopy-equivalent to $S^4$.

**8.** Let $F_n$ denote the free group on $n$ generators. Use covering space theory to prove that $F_2$ contains subgroups isomorphic to $F_n$, for every $n \geq 1$.