MATH 270ABC: Numerical Analysis

Instructor: Randolph Bank

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Qualifying Examination
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NAME ____________________

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Question 1. Let A be an $n \times n$ nonsingular matrix.

a. Prove $PA = LU$, where P is a permutation matrix, L is unit lower triangular and U is upper triangular.

b. Define the algorithm of *partial pivoting* which can be used to determine the permutation matrix P in the factorization. Using the simple 2×2 matrix,

$$A = \begin{pmatrix} \delta & 1 \\ 1 & 1 \end{pmatrix}$$

where $|\delta| \ll 1$, explain the influence of partial pivoting on the numerical stability of the $PA = LU$ factorization.

Question 2. Let

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$$

a. Compute the singular value decomposition of A.

b. Compute the generalized inverse $A^\dagger$ of A.

Question 3. Let $A = D - L - L^t$, where A is $n \times n$ symmetric, positive definite, D is diagonal, and L is strictly lower triangular.

a. Define the Jacobi, Gauss-Seidel, SOR, and SSOR iterative methods for solving $Ax = b$.

b. Assuming A has $O(n)$ nonzero entries, derive the complexity estimate $O(n \log \epsilon / \log \rho)$ to solve $Ax = b$, $x_0 = 0$ with relative error $\|x - x_k\| \leq \epsilon \|x\|$ using the Jacobi iteration. Be sure to define the parameter ρ appearing in this estimate.

Question 4. Let $f(x)$ be a vector function of a vector variable x. Assume $f(x)$ is continuous and differentiable, and that the Jacobian $J(x)$ is continuous in the ball $\mathcal{B} = \{x | \|x - x^*\| \leq \delta$ for some $\delta > 0$. More specifically, assume:

1. $f(x^*) = 0$.
2. $\|J(x)^{-1}\| \leq M$ for all $x \in \mathcal{B}$.
3. $\|J(x) - J(y)\| \leq \gamma \|x - y\|$ for all $x, y \in \mathcal{B}$.

Assume the sequence x_k is generated from a starting vector $x_0 \in \mathcal{B}$ using Newton's method without line search. Using Taylor's theorem, prove

$$\|e_{k+1}\| \leq \frac{M\gamma}{2} \|e_k\|^2$$

where $e_k = x^* - x_k$. Hint: $f(x) = f(y) + \int_0^1 J(\theta x + (1 - \theta)y)(x - y) d\theta$

Question 5. Consider the inner product (f, g), and corresponding norm $\|f\| = \sqrt{(f, f)}$ defined on a vector space $\mathcal{V}$. Let $\mathcal{S} \subset \mathcal{V}$ be a finite dimensional subspace. Let $f \in \mathcal{V}$, and let $f^* \in \mathcal{S}$ be the least squares approximation of f satisfying

$$\|f - f^*\| = \min_{v \in \mathcal{S}} \|f - v\|$$

Prove the orthogonality relation

$$(f - f^*, v) = 0$$

for all $v \in \mathcal{S}$.

Question 6. Let

$$\mathcal{I}(f) = \int_{-1}^{1} f(x)dx$$

We consider a Gaussian quadrature formula of the form

$$\mathcal{Q}(f) = w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

a. Compute the weights w_i and knots x_i to maximize the order. (Hint: use symmetry.)

b. Using the Peano Kernel Theorem, prove

$$|\mathcal{I}(f) - \mathcal{Q}(f)| \leq C_0 \|f^{vi}\|_{\infty[-1,1]}.$$

Question 7. Consider the scalar equation $y' = f(y)$ with $y(x_0) = y_0$.

- **a.** Define Euler's Method, the Backward Difference Method, and the Trapezoid Rule (Crank-Nicolson Method) for solving this equation.
- **b.** Compute the region of absolute stability for each of these methods.
- **c.** Which of these methods are A-Stable? Which are L-Stable?

Question 8. Consider the multistep formula

$$\sum_{i=0}^{p} \alpha_i y_{n-i} + h\beta_i f(y_{n-i}) = 0$$

for approximating $y' = f(y)$.

a. Define the local truncation error for this formula.

b. Define the polynomials $\rho(r)$ and $\sigma(r)$ associated with this formula.

c. Define consistency conditions for the formula in terms of ρ and σ.

d. Define the root condition for stability in terms of ρ and σ.