ALGEBRA QUALIFYING EXAM
THURSDAY SEPTEMBER 14TH

You have three hours.

There are 8 problems, and the total number of points is 80. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:____________________________________

Signature:________________________________

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total	80	

1. (10pts) Let G be a group of order $231 = (3)(7)(11)$.
(a) Show that G is isomorphic to a semidirect product $\mathbb{Z}_{77} \rtimes \mathbb{Z}_3$.

(b) Show that there are precisely two groups G of order 231 up to isomorphism.

2. (10pts) Let G be the following subgroup of 2×2 matrices over the complex numbers:

$$G = \{\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\}$$

(You don't have to show this is a group).
Prove that G has the following presentation

$$\langle a, b \mid a^4 = e,\ a^2 = b^2,\ a^{-1}ba = b^{-1} \rangle.$$

3. (10pts) (a) Carefully state Zorn's Lemma.

(b) Let R be a commutative ring and let X be any multiplicatively closed subset of R which does not contain 0. Show that R has an ideal I which is a maximal element of the collection of those ideals J such that $J \cap X = \emptyset$.

(c) If X is not empty then prove that the ideal I as in (b) must be a prime ideal.

4. (10pts) Let R be a commutative ring with 1. An R-module M is called *flat* if whenever $f\colon N \longrightarrow P$ is an injective R-linear map of R-modules then the induced map

$$M \underset{R}{\otimes} N \longrightarrow M \underset{R}{\otimes} P$$

is also injective.
If R is a PID and M is a finitely generated R-module then show that M is flat if and only if it is torsion free.

5. (10pts) Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & -y \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

where y is an indeterminate.

(a) Show that the characteristic polynomial $f(x)$ of A is irreducible in $\mathbb{Q}(y)[x]$.

(b) Show that A is diagonalisable over the algebraic closure of $\mathbb{Q}(y)$.

(c) Show that A is not diagonalisable over the algebraic closure of $\mathbb{F}_3(y)$.

6. (10pts) Let

$$\mathbb{Q} = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_n,$$

be a sequence of field extensions such that K_{i+1}/K_i is Galois of order 3 for all $0 \leq i < n$. Show that $\mathbb{Q}(\sqrt[3]{2})$ is not contained in K_n.

7. (10pts) Let $L/M/K$ be field extensions with $[L : M] < \infty$. Let A be the subfield of L consisting of all elements of L that are algebraic over K. Suppose that $M \cap A = K$.
(a) If $\alpha \in A$ and $f(x) \in M[x]$ is the minimal polynomial of α over M then show that $f(x) \in K[x]$.

(b) Now suppose, for the rest of this question, that the characteristic is zero. If $K \subset B \subset A$ is an intermediary field and $[B : K] < \infty$ then show that

$$[B : K] \leq [L : M].$$

(c) Prove that $[A : K] \leq [L : M]$.

8. (10pts) Let

$$R = \mathbb{Z}[\sqrt{-10}] = \{\, a + b\sqrt{-10} \mid a, b \in \mathbb{Z} \,\}.$$

Let $I = \langle 2, \sqrt{-10} \rangle$ be the ideal of R generated by 2 and $\sqrt{-10}$.
(a) Show that I is not a free R-module.

(b) Show that I is a projective R-module.