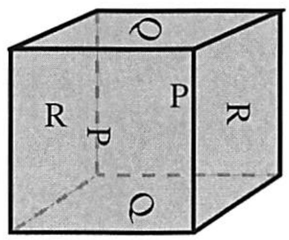
27. Summer 2017

Three-hour exam. Answer all questions; each is worth the same. You can use standard theorems, but should say when you are doing so. Please try to write good clear mathematics; merely drawing vague pictures is not enough!

1. Construct a basepointed covering space of $X = S^1 \vee S^1$ corresponding to the subgroup of the free group $\langle a, b\rangle = \pi_1(X)$ generated by the elements $aba^{-1}b^{-1}, ab^{-1}a^{-1}b, a^{-1}bab^{-1}$ and $a^{-1}b^{-1}ab$.

2. Let X be the space obtained by gluing opposite pairs of faces of a standard cube I^3 via 90 degree rotations, as shown. Compute the homology $H_*(X;\mathbb{Z})$.



3. Let Y be a space, let $f : Y \to Y$ be a self-mapping of Y, and let X be the mapping torus of f, that is, the space obtained from $Y \times I$ by identifying $(y, 1) \sim (f(y), 0)$ for each point $y \in Y$. Prove that $H_1(X;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})/\mathrm{im}\,(\mathrm{id} - f_*)$, where f_* is the induced map $H_1(Y;\mathbb{Z}) \to H_1(Y;\mathbb{Z})$.

4. Let M be a closed connected simply-connected 4-manifold. Show that $H_1(M;\mathbb{Z}) = H_3(M;\mathbb{Z}) = 0$ and that $H_2(M;\mathbb{Z})$ is a free abelian group.

5. Compute $\mathrm{Tor}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4)$.

6. Show that the Euler characteristic of a closed orientable odd-dimensional manifold is zero. Is this still true if the manifold is non-orientable?

7. Consider the natural inclusion of $V = S^1 \vee S^1$ in the torus $T = S^1 \times S^1$. Show that there does not exist a retraction $T \to V$.

8. Let M be a closed connected 3-manifold with finite fundamental group. Show that its universal cover is homotopy-equivalent to S^3.