Math 220: Fall 2013
Qualifying Exam

Friday, September 13	2:00p – 5:00p	AP&M 6402

Instructions: This is a 3-hour exam. No notes or textbooks are allowed. There are three parts to this exam. In Part I, you are asked to reiterate statements and proofs of some results from complex analysis. In Parts II and III, you are asked to solve seven problems similar to those covered in homework sets and exams in 220ABC. There are 100 points possible. Make sure to state all results and hypotheses used, and present your solutions clearly, with appropriate detail.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ denotes the open unit disk, $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$ denotes the upper half-plane, and $\mathbb{C}_- = -\mathbb{C}_+$ denotes the lower half-plane.

Part I

1. (*30 points*) Choose 4 of the following theorems, and write out their statements carefully and completely (*5 points each*). From among these, choose 2, and sketch their proofs (*5 points each*).

the Open Mapping Theorem	the Poisson Integral Formula
Rouché's Theorem	the Schwarz Reflection Principle
Morera's Theorem	the Argument Principle
Runge's Theorem	the Great Picard Theorem
the Riemann Mapping Theorem	the Monodromy Theorem

Part II Do all four of the following problems.

2. (*10 points*) Use residues to calculate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2}\, dx$.

3. (*10 points*) A *fixed point* z of a function f is a point where $f(z) = z$; i.e. $f(z) - z = 0$. Let f be holomorphic on a neighborhood of $\overline{\mathbb{D}}$. Suppose that $\sup_{z \in \partial\mathbb{D}} |f(z)| \leq 1$, and that f has no fixed points on $\partial\mathbb{D}$. Prove that f has exactly one fixed point in $\mathbb{D}$.

4. (*10 points*) Find a holomorphic function that maps the upper half-plane $\mathbb{C}_+$ *onto* the punctured unit disk $\mathbb{D} \setminus \{0\}$.

5. (a) (*5 points*) Prove that the function $f(z) = \left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2$ is a conformal bijection mapping $\mathbb{C}_-$ onto $\mathbb{D} \setminus [0, 1)$.

(b) (*5 points*) Let $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{R}$ and $ad - bc < 0$. Show that f is a conformal bijection from $\mathbb{C}_+$ onto $\mathbb{C}_-$.

Part III Do at least 3 of the following 4 problems.

6. *(10 points)* Let $\varphi\colon \mathbb{D} \setminus [0,1) \to \mathbb{D}$ be a conformal bijection. Prove that φ does not extend to a homeomorphism $\overline{\mathbb{D}} \to \overline{\mathbb{D}}$.

7. *(10 points)* Suppose $f \in \mathrm{Hol}(\mathbb{C}_+)$ has the property that, for any sequence z_n in $\mathbb{C}_+$ that converges in $(0,1)$, $f(z_n) \to 0$. Prove that $f \equiv 0$ on $\mathbb{C}_+$.

8. *(10 points)* Let $U \subseteq \mathbb{C}$ be a domain, and let $z_0 \in U$. For $M > 0$, define $\mathscr{F}_M$ to be the family of functions in $\mathrm{Hol}(U)$ with the properties that

$$|f(z_0)| \le M, \qquad \mathrm{Re} f(z) > 0 \quad \forall z \in U.$$

Show that $\mathscr{F}_M$ is a normal family: i.e. it is relatively compact in the topology of uniform convergence on compact subsets of U.

9. **(a)** *(5 points)* Does there exist a harmonic function u on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, so that $u(e^{i\theta}) = \cos^2\theta$ for $\theta \in [0, 2\pi)$? If not, prove it. If so, what is the value $u(0)$?

   (b) *(5 points)* Prove that there is no holomorphic function f on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, so that $f(e^{i\theta}) = e^{-i\theta}$ for $\theta \in [0, 2\pi)$.