ALGEBRA QUALIFYING EXAM
TUESDAY MAY 23RD

You have three hours.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total	70	

There are 7 problems, and the total number of points is 70.

Name: ______________________________

Signature: ______________________________

1. (10pts) Let H be a finite Abelian group, with product written multiplicatively. Let $\mathbb{Z}_2 = \{ e, a \}$, also written multiplicatively, so that $a^2 = e$. The corresponding *generalized dihedral group* is the semidirect product $G = H \rtimes_\phi \mathbb{Z}_2$, where $\phi(a)$ is the automorphism of H given by inverting elements, that is, $[\phi(a)](h) = h^{-1}$.

(a) Suppose that $H = \mathbb{Z}_3 \times \mathbb{Z}_3$. Find a presentation of $G = H \rtimes_\phi \mathbb{Z}_2$, with a brief justification.

(b) For which Abelian groups H is $G = H \rtimes_\phi \mathbb{Z}_2$ also Abelian?

(c) Must $G = H \rtimes_\phi \mathbb{Z}_2$ be solvable?

2. (10pts) Suppose that G is a simple group of order $|G| = 168 = 2^3 \cdot 3 \cdot 7$. (There really is a simple group of that order).

(a) Show that G does not have any subgroup H with index $[G : H] \leq 6$.

(b) Show that G does have a subgroup H with index $[G : H] = 8$.

3. (10pts) Let $R = \mathbb{Z}[\sqrt{-6}]$.
(a) Prove that 3 is an irreducible element of R which is not prime.

(b) Find an element $r \in R$ such that the ideal $I = \langle 3, r \rangle$ is a maximal ideal in R. Prove that I is not a principal ideal.

4. (10pts) Let R be a commutative ring, with unity, not equal to zero. If M and N are R-modules, (for the purposes of this question) call a multilinear map

$$f\colon M \times M \times M \longrightarrow N,$$

cyclically trilinear if

$$f(a, b, c) = f(b, c, a) \qquad \text{for all} \qquad a, b, c \in M.$$

Show how to construct a universal cyclically trilinear map

$$u\colon M \times M \times M \longrightarrow C_3(M),$$

so that u is cyclically trilinear and if

$$f\colon M \times M \times M \longrightarrow N$$

is any other cyclically trilinear map, then there is a unique linear map

$$\phi\colon C_3(M) \longrightarrow N.$$

5. (10pts) Let J be an $n \times n$ Jordan block with eigenvalue $\lambda \in \mathbb{C}$.
(a) Suppose that $\lambda \neq 0$. Show that the Jordan canonical form of J^2 is an $n \times n$ Jordan block with eigenvalue λ^2.

(b) Suppose that $\lambda = 0$. Show that the Jordan canonical form of J^2 has two Jordan blocks with eigenvalue 0; if n is even these Jordan blocks have size $n/2 \times n/2$ and if n is odd one Jordan block has size $(n+1)/2 \times (n+1)/2$ and the other has size $(n-1)/2 \times (n-1)/2$.

6. (10pts) Let F be a prime field, so that F is either isomorphic to $\mathbb{Q}$ or $\mathbb{F}_p$, p a prime. Show that the algebraic closure of F is infinite dimensional over F.

7. (10pts) Let $L/\mathbb{Q}$ be a splitting field for the polynomial $x^{11}-1$. Find all intermediary fields $L/M/\mathbb{Q}$. For each intermediary field M find an element $\alpha \in M$ such that $M = \mathbb{Q}(\alpha)$.