

Algebra qualifying exam, Spring 2015

Instructions: Do as many of the problems as well as you can; the exam may be too long for you to finish. You may quote major theorems from the textbook or class, unless the point of the problem is reproduce the proof of such a result. In multistep problems, you may use the result of part (a) in the proof of part (b) even if you do not know how to do part (a).

1 (15 pts). Let G be a group of order p^2q^2 , where p and q are primes with $p > q$.

(a) (7 pts). Prove that either $|G| = 36$ or else G has a normal Sylow p -subgroup.

(b) (8 pts). Prove that G is solvable. (Hint: when $|G| = 36$, consider the action of G on the left cosets of a Sylow 3-subgroup).

2 (15 pts).

(a) (10 pts). Let K/F be a field extension and let $f \in F[x]$. If $A = F[x]/(f)$, show that $K \otimes_F A \cong K[x]/(f)$ as F -algebras.

(b) (5 pts). Again let K/F be a field extension, assume that $\text{char } F = 0$, and let $\alpha, \beta \in K$ be algebraic over F . Let $F_1 = F(\alpha)$ and $F_2 = F(\beta)$. Consider the F -algebra $R = F_1 \otimes_F F_2$. Show that R is isomorphic as a ring to a direct product of finitely many fields.

3 (15 pts). Let $f(x) = x^6 - 3$. Let K be the splitting field of $f(x)$ over \mathbb{Q} . Show that $\text{Gal}(K/\mathbb{Q})$ is isomorphic to a dihedral group.

4 (10 pts). Let R be an integral domain. Prove that R is a PID if and only if every submodule of a finitely generated free R -module is again free.

5 (10 pts). Let R be a commutative ring, and let S be a multiplicative system in R . Let $S^{-1}R$ be the localization of R at S and let $\phi : R \rightarrow S^{-1}R$ be the canonical homomorphism sending $r \mapsto \frac{r}{1}$. For an ideal I of R we write I^e for the ideal of $S^{-1}R$ generated by $\phi(I)$, and for an ideal J of $S^{-1}R$ we write J^c for the ideal of R given by $\phi^{-1}(J)$.

(a) (6 pts). Prove that for any ideal J of $S^{-1}R$, one has $(J^c)^e = J$, while for any ideal I of R , one has $(I^e)^c = \{x \in R \mid sx \in I \text{ for some } s \in S\}$.

(b) (4 pts). Show that if R is noetherian, then $S^{-1}R$ is noetherian. Give an example which shows that the converse need not hold.

6 (10 pts). Recall that a subgroup H of K is *characteristic* in K , written $H \text{ char } K$, if for every automorphism $\sigma \in \text{Aut}(K)$, we have $\sigma(H) = H$. Recall also that the notation $K \trianglelefteq G$ means that K is a normal subgroup of G .

(a) (5 pts). Prove that if $H \text{ char } K$ and $K \trianglelefteq G$, then $H \trianglelefteq G$.

(b) (5 pts). Prove that if G is a finite group with Sylow- p subgroup P for some prime p , and $K = N_G(P)$ is the normalizer of P in G , then $N_G(K) = K$.

7 (10 pts). Let $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in M_3(F)$, where F is an algebraically closed field.

Find both the Jordan canonical form and rational canonical form of M . (The answer may depend on the characteristic of F).

8 (10 pts). Let K be a field with $G \subseteq \text{Aut}(K)$ a finite group of automorphisms of K . Let $F = \text{Fix}(G)$. Let $\alpha \in K$ and let $f = \text{minpoly}_F(\alpha)$. Let $H = \{g \in G \mid g(\alpha) = \alpha\}$ and fix $g_1, g_2, \dots, g_m \in G$ such that g_1H, \dots, g_mH are the distinct left cosets of H in G .

(a) (7 pts). Show that $f(x) = (x - g_1(\alpha)) \dots (x - g_m(\alpha))$. (Hint: show the polynomial on the right has coefficients in F .)

(b) (3 pts). Use part (a) to conclude that the field extension K/F is separable and normal.