Algebra Qualifying Exam
Spring 2016

Instructions: Do as many problems as you can, as completely as you can. You may attempt the problems in any order, but make it clear which problem you are working on.
If a problem has multiple parts, you may use the result of any part (even a part you do not solve) in the proof of another part of that problem. You may quote theorems proved in class or in the textbook, unless the point of the problem is to reproduce the proof of that theorem.

Problem 1 [10 points] Let p be a prime number and let G be a p-group. Prove that G is nilpotent.

Problem 2 [10 points] Let G be a (not necessarily finite) group with $|G| > 2$. Prove that there is an automorphism $\varphi : G \to G$ other than the identity map.

Problem 3 [10 points] Construct a non-abelian group of order 39 or prove that no such group exists.

Problem 4 [10 points] Let K be a field.

(a) Is the polynomial ring $K[x, y]$ Noetherian? Why or why not?
(b) Find a subring $S \subseteq K[x, y]$ which is *not* Noetherian.

Problem 5 [10 points] Let K and L be fields of orders 9 and 27, respectively. Is K isomorphic to a subfield of L?

Problem 6 [15 points] Is the equation $x^5 - 16x + 2 = 0$ solvable in radicals?

Problem 7 [10 points] Prove that there exists a Galois extension of $\mathbb{Q}$ whose Galois group is a cyclic group of order 4.

Problem 8 [15 points] Let $R = \mathbb{Z}[\sqrt{-6}\,] = \{a + b\sqrt{-6}\,; a, b, \in \mathbb{Z}\}$. Let $I = (2, \sqrt{-6}\,)$ be the ideal of R generated by 2 and $\sqrt{-6}$.

(a) Show that I is not a free R-module.
(b) Show that I is a flat R-module.

Problem 9 [10 points] Let R be a commutative ring with unit, M be an R-module and I an ideal of R. Suppose that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of R that contain I. Prove that $M = IM$.