Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiners: Randolph Bank and Philip E. Gill

9:00am–12:00pm
Friday May 21, 1999

NAME ____________________

#1.1	20	
#1.2	20	
#1.3	20	
#2.1	20	
#2.2	20	
#2.3	20	
#3.1	20	
#3.2	20	
Total	160	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Linear Equations and Linear Least Squares

Question 1.1. Let $\hat{x}$ be an approximate solution of $Ax = b$, where A is a nonsingular $n \times n$ matrix, $n > 1$.

(a) Prove that an infinite number of matrices F satisfy the relation $(A + F)\hat{x} = b$.

(b) Find a matrix E that has the smallest two-norm among all matrices F satisfying $(A + F)\hat{x} = b$.

(c) Show that for the matrix E of part (b),

$$\frac{\|r\|}{\|b\|} \geq \frac{\|E\|/\|A\|}{1 + \|E\|/\|A\|},$$

where all norms are two-norms. Hence show that if

$$\frac{\|r\|}{\|b\|} \leq \epsilon < 1, \quad \text{then} \quad \frac{\|E\|}{\|A\|} \leq \frac{\epsilon}{1 - \epsilon}.$$

Briefly state the significance of this result when solving linear systems using Gaussian elimination.

Question 1.2.

(a) Let u denote the unit roundoff and assume that $nu < 1$ for some positive integer n. If $|\delta_i| \leq u$, show that

$$\prod_{i=1}^{n}(1 + \delta_i) = 1 + \theta_n,$$

where $|\theta_n| \leq \gamma_n = nu/(1 - nu)$.

(b) Consider the matrix-vector product $y = Ax$ for $A \in I\!R^{m \times n}$. Assuming the standard model for floating-point computation, let $\hat{y}$ denote the computed value of y when A and x are representable. Derive the following:

- **(i)** A bound on the Frobenius norm of the absolute backward error in $\hat{y}$, assuming that A is data and x is exact.
- **(ii)** A bound on the two-norm of the absolute forward error in $\hat{y}$.
- **(iii)** A bound on the two-norm of the relative forward error in $\hat{y}$, assuming that A is nonsingular.

Question 1.3. Assume that $A \in I\!R^{m \times n}$.

(a) Derive the necessary and sufficient condition for a vector to be a solution of minimum two-norm for the compatible system $Ax = b$.

(b) Show that the least-length solution of the compatible system $Ax = b$ is unique.

(c) For any x, let r denote the residual vector $b - Ax$. Derive the necessary and sufficient conditions for a vector to solve the problem $\min \|b - Ax\|_2$. Discuss the circumstances under which the least-squares solution is unique.

(d) If $A \in I\!R^{n \times n}$ and $x \in I\!R^n$ are given, find the value λ^* that minimizes the function $\|Ax - \lambda x\|_2$.

2. Eigenvalues and Singular Values

Question 2.1. Prove the existence of the singular-value decomposition for an $m \times n$ *complex-valued* matrix with $m \geq n$.

Question 2.2. Assume that $A \in \mathbb{C}^{n\times n}$. Given a positive scalar ϵ and $E \in \mathbb{C}^{n\times n}$ such that $\|E\|_2 = 1$, let λ denote an eigenvalue of $A + \epsilon E$.

(a) Show that if λ is not an eigenvalue of A, then λ lies in the domain

$$\frac{1}{\|(A - \lambda I)^{-1}\|_2} \leq \epsilon.$$

(b) Show that if E is of the form $-pq^H$, then the value of ϵ that produces the eigenvalue λ is

$$\epsilon = \frac{1}{q^H(A - \lambda I)^{-1}p}.$$

(c) Hence find a perturbation that gives an eigenvalue on the boundary of the domain defined in part (a).

Question 2.3. Consider the unshifted QR method for finding the eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$. If R_k and Q_k are the matrices generated at iteration k, show that

(a) $A_{k+1} = Q_k^H A_k Q_k$.

(b) $A_{k+1} = Q_k^H Q_{k-1}^H \cdots Q_0^H A Q_0 Q_1 \cdots Q_k$.

(c) $A^{k+1} = \widehat{Q}_k \widehat{R}_k$, where $\widehat{R}_k = R_k R_{k-1} \cdots R_1 R_0$ and $\widehat{Q}_k = Q_0 Q_1 \cdots Q_k$.

(d) Define an unshifted QR method that requires $O(n^2)$ floating-point operations each iteration.

3. Interpolation, Approximation and ODEs

Question 3.1. Let $f \in C^2(I)$, $I = [a,b]$, and let $x_i = a + ih$, $0 \le i \le n$, $h = (b-a)/n$ be a uniform mesh on I. Let $\mathcal{S}$ be the space of continuous piecewise linear polynomials with respect to this uniform mesh and let $\tilde{f}$ denote the continuous piecewise linear polynomial interpolant of f.

(a) Compute the dimension of $\mathcal{S}$ and define the standard *nodal basis* functions $\{\phi_i\}$ for $\mathcal{S}$.

(b) Using the Peano Kernel Theorem, prove:

$$\|f - \tilde{f}\|_{\mathcal{L}^2(I)} \le Ch^2 \|f''\|_{\mathcal{L}^2(I)}$$

(You do NOT need to explicitly evaluate the constant C.)

Question 3.2. Let

$$\mathcal{I}(f) = \int_{-1}^{1} f(x)dx$$

Consider the two point Gauss-Legendre quadrature formula of the form

$$\mathcal{Q}(f) = w_1 f(x_1) + w_2 f(x_2) \tag{3.1}$$

(a) Find the knots x_1 and x_2 and the weights w_1 and w_2 for the Gauss-Legendre formula (3.1).

(b) What is the form of the error $\mathcal{I}(f) - \mathcal{Q}(f)$? Be sure to explicitly evaluate the constant.

(c) Write down the composite formula for approximating

$$\int_a^b f(x)dx$$

on a uniform mesh of size h (note here the reference interval is $[-1,1]$).

(d) Write down an expression for the error in the composite formula.