

Name: _____ Student ID#: _____

Problems/Page Numbers	Total Points	Your Score
Problem 1 / Page 2	40 Points	
Problem 2 / Page 4	20 Points	
Problem 3 / Page 5	20 Points	
Problem 4 / Page 6	20 Points	
Problem 5 / Page 7	20 Points	
Problem 6 / Page 8	20 Points	
Problem 7 / Page 9	20 Points	
Problem 8 / Page 10	20 Points	
Total: 8 Problems / 10 Pages	Total: 180 Points	

General Instructions

1. This is a three-hour, closed-book, closed-note, and no-calculator exam. There are 8 problems of total 180 points.
2. Be sure to carefully motivate all (nontrivial) claims and statements. Be sure also to clearly explain and justify your answers.
3. You may cite without proof any results proved in the text or in the class, as long as they are not what the problem explicitly asks you to prove. You may also use the results of prior problems or prior parts of the same problem when solving a problem.
4. If you cite a result (a theorem, lemma, etc.) that is proved in the text or class, refer to it either by name (if it has one) or explain clearly what it states and verify explicitly all the hypotheses.

Notation: m is the Lebesgue measure.

Problem 1 (40 points). Determine if the statements below are **True** or **False**. If **True**, give a brief proof. If **False**, give a counterexample or prove your assertion in a different way as you prefer. If your claim follows from a theorem in the text, name the theorem or describe it otherwise, and explain carefully how the conclusion follows.

(a) (5 points) If the derivative of a real-valued, absolutely continuous function f on \mathbb{R} vanishes almost everywhere (with respect to the Lebesgue measure), then f must be a constant on \mathbb{R} .

(b) (5 points) If X is a locally compact Hausdorff topological space and $x \in X$, then the Dirac delta measure δ_x is a Radon measure on Borel sets of X . (Recall that δ_x is defined by $\delta_x(E) = 1$ if $x \in E$ and $\delta_x(E) = 0$ if $x \notin E$ for any Borel set $E \subseteq X$.)

(c) (5 points) Let H be a real Hilbert space. Assume a sequence $\{x_n\}$ converges to x weakly in H . Then $\lim_{n \rightarrow \infty} \|x_n\| = \|x\|$ implies that $\{x_n\}$ converges to x in H .

(d) (5 points) Let f, g , and f_n and g_n ($n = 1, 2, \dots$) be all real-valued, Lebesgue measurable functions on \mathbb{R} . If $f_n \rightarrow f$ and $g_n \rightarrow g$ in measure, then $f_n g_n \rightarrow fg$ in measure.

Problem 2 (20 points). Let (X, M, μ) be a measure space and $f \in L^1(\mu)$. Prove that

$$\lim_{t \rightarrow \infty} t\mu\{x \in X : |f(x)| \geq t\} = 0.$$

Problem 3 (20 points). Let $f : [0, 1] \rightarrow \mathbb{R}$ be a continuous function. Define the signed Borel measure μ on $[0, 1]$ by $d\mu = f dm$. Assume

$$\int_{[0,1]} x^n d\mu = 0, \quad n = 0, 1, 2, \dots$$

Prove that $\mu = 0$.

Problem 4 (20 points). Let X and Y be two Banach spaces and denote by $L(X, Y)$ the space of all continuous linear operators from X to Y . Let $A_n \in L(X, Y)$ ($n = 1, 2, \dots$). Assume that $\lim_{n \rightarrow \infty} A_n(x)$ exists for each $x \in X$. Define $A(x) = \lim_{n \rightarrow \infty} A_n(x)$. Prove that $A \in L(X, Y)$.

Problem 5 (20 points). Let (X, \mathcal{M}, μ) be a measure space. Assume that $f : X \rightarrow \mathbb{R}$ and $f_n : X \rightarrow \mathbb{R}$ ($n = 1, 2, \dots$) are all μ -measurable and that $f_n \rightarrow f$ in measure. Let p be a real number such that $1 \leq p < \infty$. Assume there exists $g \in L^p(\mu)$ such that $|f_n| \leq g$ on X for all $n \geq 1$. Prove that all f and f_n ($n = 1, 2, \dots$) are in $L^p(\mu)$ and that $f_n \rightarrow f$ in the $L^p(\mu)$ -norm.

Problem 6 (20 points). The problem concerns finding an explicit solution to the equation $-\Delta u + u = f$ for $f \in C_c^\infty(\mathbb{R}^n)$, using the Fourier transform $\hat{\phi}(z) \doteq \int e^{-2\pi\langle x,z \rangle} \phi(x) dm(x)$ and its inverse.

(a) Assume that the solution u and all its first-order and second-order partial derivatives $\frac{\partial u}{\partial x_i}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ ($1 \leq i, j \leq n$) are in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Prove

$$\hat{u}(z) = \frac{\hat{f}(z)}{1 + 4\pi^2|z|^2}.$$

(b) Prove, via the identity $(1 + 4\pi^2|z|^2)^{-1} = \int_0^\infty e^{-t(1+4\pi^2|z|^2)} dt$, that

$$u(x) = \int_0^\infty \frac{e^{-t}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) dm(y) dt.$$

Problem 7 (20 points). Let $X = [-\pi, \pi]$ and consider the Lebesgue measure. Let p be a real number with $1 \leq p < \infty$. Define for each integer $k \geq 1$ that $f_k(x) = \sin(kx)$ ($x \in X$).

- (a) Prove that the sequence $\{f_k\}$ converges weakly to 0 in $L^p(X)$.
- (b) Prove that the sequence $\{f_k\}$ does not converge to 0 strongly in $L^p(X)$.

Problem 8 (20 points). Let f be a monotonically non-decreasing function on \mathbb{R} . Prove that its distributional derivative is a Borel measure.

(Hint: You may apply the Riesz-Markov representation theorem. In doing so, you need to state the theorem clearly and verify the assumptions carefully.)