SPRING 2013 UCSD ALGEBRA QUALIFYING EXAM

Instructions: Do as many problems as you can, as completely as you can. If a problem has multiple parts, you may use the result of any part (even a part you do not solve) in the proof of another part of that problem. You may quote theorems proved in class or in the textbook, unless the point of the problem is to reproduce the proof of that theorem.

(1) (10 pts) Does there exist a non-Abelian group of order $2013 = (3)(11)(61)$? Justify your answer.

(2) (10 pts) Let A be a commutative ring. Prove that any projective A-module is flat. (Hint: first prove it for free modules).

(3) (10 pts) Let A be a PID and let M be a torsion finitely generated $A-$module. Suppose that $p \in A\backslash\{0\}$ is a prime element such that p^k is one of the elementary divisors of M for some $k \geq 1$. Show that M has a submodule which is isomorphic to $A/(p)$ as an A-module.

(4) (10 pts) Let A be an integral domain and let F be its field of fractions. For any nonzero element f of A, let $A[f^{-1}]$ be the subring of F generated by A and f^{-1}. Suppose that the ideal generated by $f_1, \ldots, f_k \in A \setminus \{0\}$ is the entire ring A, i.e. $(f_1, \ldots, f_k) = A$.
  (a) (3 pts) Prove that for any positive integers $n_1, \ldots, n_k$ we have that
  $$(f_1^{n_1}, \ldots, f_k^{n_k}) = A.$$
  (b) (7 pts) Prove that $A = A[f_1^{-1}] \cap A[f_2^{-1}] \cap \cdots \cap A[f_k^{-1}]$. (Hint: for x in the right hand side consider $\{a \in A |\ ax \in A\}$).

(5) (15 pts) Let g be a torsion element of $\mathrm{GL}_n(\mathbb{Q})$, i.e. $g^m = 1$ for some positive integer m. Let us assume m is the order of g, i.e. $g^{m'} \neq 1$ for $0 < m' < m$.
  (a) (8 pts) Prove that g is diagonalizable over $\mathbb{C}$. (Hint: think about the minimal polynomial of g and its Jordan form.)
  (b) (7 pts) Prove that there is a positive number M depending on n such that the order of any torsion element $g \in \mathrm{GL}_n(\mathbb{Q})$ is at most M. (Hint: think about the eigenvalues of g and field theory.)

(6) (15 pts) Let E be the splitting field of $x^p - 2$ over $\mathbb{Q}$, where p is an odd prime.
  (a) (7 pts) Prove that $E = \mathbb{Q}[\zeta_p, \sqrt[p]{2}]$ and find $[E : \mathbb{Q}]$. (Here ζ_p is a primitive pth root of unity.)
  (b) (8 pts) Find $\{(\sigma(\zeta_p), \sigma(\sqrt[p]{2}))|\ \sigma \in \mathrm{Gal}(E/\mathbb{Q})\}$. Justify your answer.

(7) (10 pts) Let F be a field and p be a prime. Suppose that the degree of any finite field extension E/F is divisible by p. Prove that the degree of any finite separable extension E/F is a power of p.

(8) (10 pts)
  (a) (7 pts) Give an example of an integral domain which is not a UFD. Give a brief proof that it is not a UFD.
  (b) (3 pts) Find a UFD A and a prime ideal $\mathfrak{p} \subseteq A$ such that $A/\mathfrak{p}$ is not a UFD. (Hint: use part (a)!)