MATH 240 QUALIFYING EXAM: SPRING 2011

Instructions: Please answer all 5 questions. Partial credit will be given, so you should attempt all problems. You may quote any standard proposition/theorem from Folland or Lieb-Loss, but **not** the homeworks. **If you use a major result like MCT etc, you need to state the theorem you are using *explicitly*.** In this exam dx is integration with respect to Lebesgue measure, and $|E|$ denotes the Lebesgue measure of E.

I) **True/False, Short Answer:** Please answer True or False. You must **support your claim** with a short explanation or counterexample, but you need not give an entire proof:

a) (5 pts) If $T_n \in \mathcal{L}(X,Y)$ is a sequence of bounded linear operators with X, Y are Banach spaces, then if $Tx := \lim_n T_n x$ exists in the Y norm for all $x \in X$, one has $T \in \mathcal{L}(X,Y)$.

b) (5 pts) Let $(X, \mathcal{M}, \mu)$ be any measure space. If $f_n, f \in L^1(d\mu)$ are measurable functions such that $f_n \to f$ μ-a.e. and $\lim \int f_n \to \int f$, then $f_n \to f$ in $L^1(d\mu)$.

c) (5 pts) If $\{X_\alpha\}_{\alpha \in A}$ is any collection of compact Hausdorff topological spaces, then $E \subset \Pi_{\alpha \in A} X_\alpha$ with the product topology is compact iff it is closed.

d) (5 pts) If $f_n \in L^2([0,1])$ (Lebesgue measure) converge weakly to $f \in L^2([0,1])$, then there is a subsequence such that $f_{n_k} \to f$ pointwise a.e. with respect to Lebesgue measure.

e) (5 pts) If μ is positive and finite Borel measure on $\mathbb{R}^n$ which is a.c. with respect to Lebesgue measure, then $\lim_{r\to 0} \frac{\mu(B_r(x))}{|B_r(x)|} \to 0$ for a.e. $x \in \mathbb{R}^n$ implies $\mu \equiv 0$. Here $B_r(x)$ is the ball of radius r at x.

II) For each integer $k > 0$ denote by $\Delta_k(j) = [j2^{-k}, (j+1)2^{-k}]$ where $j \in \mathbb{Z}$, the dyadic rational interval of length 2^{-k} starting at $j2^{-k}$. Let $f \in L^1(\mathbb{R})$, and define:

$$A_k(f)(x) \ = \ \sum_j a_k(j) \mathbf{1}_{\Delta_k(j)}(x) \ , \qquad a_k(j) \ = \ \frac{1}{|\Delta_k(j)|} \int_{\Delta_k(j)} f(y) dy \ .$$

a) (5 pts) Prove that $\| A_k f \|_{L^1(\mathbb{R})} \leqslant \| f \|_{L^1(\mathbb{R})}$ for all $k > 0$, $f \in L^1(\mathbb{R})$.

b) (10 pts) Show that if $f \in L^1(\mathbb{R})$ then $A_k(f) \to f$ in $L^1(\mathbb{R})$ as $k \to \infty$. (Hint: First try to show this when f is continuous and compactly supported).

III) (15 pts) Let $1 \leqslant p, p' \leqslant \infty$ be fixed, with p, p' dual indices. Suppose $K(x,y) \geqslant 0$ is a Lebesgue measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ such that there exists a constant $0 \leqslant K_0 < \infty$ with:

$$\iint g(x) K(x,y) f(y) dx dy \leqslant K_0 \| g \|_{L^{p'}(\mathbb{R}^n)} \| f \|_{L^p(\mathbb{R}^n)} \ ,$$

for all measurable $f, g \geqslant 0$ on $\mathbb{R}^n$. Show that if $f \in L^p(\mathbb{R}^n)$, then the function $Kf(x) := \int_{\mathbb{R}^n} K(x,y) f(y) dy$ is well defined for (Lebesgue) a.e. $x \in \mathbb{R}^n$, and one has $\| Kf \|_{L^p(dx)} \leqslant K_0 \| f \|_{L^p(dx)}$.

IV) Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

a) (10 pts) Let $1 \leqslant p \leqslant \infty$. Show that for fixed $M > 0$ the ball $\| f \|_{L^p(d\mu)} \leqslant M$ is closed in $L^1(d\mu)$.

b) (10 pts) Show that the whole of $L^p(d\mu)$ is closed in $L^1(d\mu)$ iff there exists $C > 0$ such that $\| f \|_{L^p(d\mu)} \leqslant C \| f \|_{L^1(d\mu)}$ for all $f \in L^p(d\mu)$.

c) (10 pts) Now let $1 < p \leqslant \infty$. Show that the assumptions of part b) above holds iff both $L^p(d\mu)$ and $L^1(d\mu)$ are finite dimensional. (Hint: Show that if $(X, \mathcal{M}, \mu)$ is any measure space where there exists $0 < c, C < \infty$ such that $c \leqslant \mu(E) \leqslant C$ for every set $E \in \mathcal{M}$ of nonzero measure, then $L^1(d\mu)$ is finite dimensional).

V) (15 pts) Let $f \in L^2([0, 2\pi])$, and set $S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$ to be the N^{th} symmetric partial sum of its Fourier series. Here we are defining $\hat{f}(n) = (2\pi)^{-1} \int_0^{2\pi} e^{-inx} f(x) dx$. Show that there exists a subsequence $N_k \to \infty$ so that $S_{N_k} f \to f$ a.e. with respect to Lebesgue measure in $[0, 2\pi]$.