281ABC Qualifying Examination, Spring 2008

Problem 1

Let $X_1, \ldots, X_n$ be iid Exp(θ).

1. Identify the exact distribution of $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.
2. Find a $(1-\alpha)100\%$ UMA upper confidence bound for θ by inverting the appropriate UMP one-sided test.
3. Identify the large-sample distribution of $\bar{X}$.
4. Find a variance stabilizing transformation for $\bar{X}$, and call this $h(\cdot)$.
5. Find an approximate $(1-\alpha)100\%$ upper confidence bound for θ from the large-sample distribution of $h(\bar{X})$.
6. Which of the two confidence bounds would you prefer (the one from part 2 or part 5) and why?
7. Is the transformation $h(\cdot)$ also normalizing? If not, can you find a normalizing transformation? [HINT: try a power-law transformation, and use the fact that

$$E[\sqrt{n}(h(\bar{X}) - h(\theta))]^3 = \left([h'(\theta)]^3\mu_3 + 3h''(\theta)[h'(\theta)]^2\mu_2^2\right)/\sqrt{n} + O(1/n)$$

where μ_k is the kth central moment of X_1.]

Problem 2

1. Let $X_1, \ldots, X_n$ be iid from a strictly increasing, continuous cdf F, and let $Y_i = F(X_i)$ for $i = 1, \ldots, n$. Show that the common distribution of the Y_i is Uniform (0,1).

2. Can you relax the strictly increasing assumption to just continuity of F in part (a)?

3. Use the result of part (a) to show that the P-value of a general test of a point null hypothesis is uniformly distributed under the null. You may assume that the test is based on a test statistic T that has a strictly increasing, continuous cdf F. Suppose also (for simplicity) that the rejection region is of the type $T >$ some threshold t.

Problem 3

Let $X_1, \ldots, X_n$ be i.i.d $U(\xi - \theta, \xi + \theta)$, where $\xi \in \mathbb{R}$ and $\theta > 0$ are both unknown.

1. Show that $X_{(1)}$ and $X_{(n)}$ are sufficient statistics and find their (marginal) distributions. [Bonus if you find the joint distribution.]

2. Let $Y = (X_{(n)} - X_{(1)})/2$, with density $\psi_\theta(y) = 1/\theta \; \psi(y/\theta)$, where

$$\psi(y) = n(n-1)(1-y)y^{n-2} \quad \text{on } y \in (0, 1)$$

   [Bonus if you prove it.] Show that $\psi_\theta, \theta > 0$ has the monotone likelihood ratio property.

3. For fixed $\delta > 0$, consider testing $H : \theta \leq \delta$ versus $K : \theta > \delta$. Show the problem is invariant under the group of transformations

$$(x_1, \ldots, x_n) \to (x_1 + c, \ldots, x_n + c), c \in \mathbb{R}$$

   and that $X_{(1)}$ and $X_{(n)}$ are equivariant.

4. Find the UMPI level α test (be as explicit as possible).

Problem 4

Let $X_1, \ldots, X_m; Y_1, \ldots, Y_n$ be independent with

$$X_i \sim \mathcal{N}(\xi, \sigma_i^2), \qquad Y_j \sim \mathcal{N}(\eta, \tau_j^2)$$

The parameters $\sigma_1, \ldots, \sigma_m; \tau_1, \ldots, \tau_n$ are *known* positive constants satisfying

$$\sum_{i=1}^{m} \frac{1}{\sigma_i^2} = \sum_{j=1}^{n} \frac{1}{\tau_j^2}$$

Consider testing $H : \eta \le \xi$ versus $K : \xi > \eta$.

1. Show that the following are sufficient statistics and find their joint distribution:
$$U = \sum_{i=1}^{m} \frac{X_i}{\sigma_i^2}, \qquad V = \sum_{j=1}^{n} \frac{Y_j}{\tau_j^2}$$

2. Find the UMP level α test (be as explicit as possible).
(Hint: for a particular alternative $\xi_1 > \eta_1$, the distribution assigning probability one to $\xi = \eta = (\xi_1 + \eta_1)/2$ is least favorable.)