2006-2007 Topology Qual

Three hour exam. Each question is worth five marks. You can use any standard facts or theorems in your work provided you state clearly that you are doing so. Please try to write good clear mathematics!

1. Consider a solid tetrahedron $ABCD$. The face ABC is glued to ABD by an affine map preserving the order of vertices (i.e. A goes to A, B goes to B, C goes to D.) Similarly, BCD is glued to ACD. Compute the fundamental group of the resulting quotient space.

2. Let X_n be the bouquet of n circles, whose fundamental group (based at the vertex of the bouquet) is the free group F_n on n generators.

(i). Draw a covering of X_3 by X_5. Find the subgroup of $F_3 =< a, b, c >$ to which your cover corresponds under the correspondence between subgroups of F_3 and based connected covers of X_3.

(ii). Show that X_4 cannot cover X_3.

3. Compute the integral homology $H_*(\mathbb{R}P^2 \times \mathbb{R}P^3; \mathbb{Z})$.

4. Let $T \subseteq S^4$ be a (perhaps knotted) subspace homeomorphic to the 2-torus. Let N be a closed regular neighbourhood of T, so that N is homotopy equivalent to T. Let X be S^4 minus the interior of N, so that X is a compact 4-manifold with boundary. By considering the relative cohomology $H^*(S^4, N)$ and applying excision and Lefschetz duality, calculate the homology of X.

5. On any closed surface Σ_g of genus $g \geq 1$, it is possible to find a pair of simple closed curves (submanifolds homeomorphic to S^1) meeting transversely once. Use this fact together with intersection theory to show that any map $S^2 \to \Sigma_g$ has degree zero.

6. Use the Hurewicz theorem to calculate $\pi_3(\mathbb{R}P^3 \vee S^3)$.

7. Show that any closed (i.e. compact, without boundary) 6-manifold which is 2-connected (i.e is path-connected, simply-connected and has $\pi_2 = 0$) must have even Euler characteristic.

8. Let M^3 be a *homology sphere* – a closed 3-manifold having the same homology groups as S^3 – and let $X = \Sigma M$ be its suspension. What are the fundamental group and homology groups of X? Show that X is homotopy equivalent to S^4.