Ph.D./Masters Qualifying Examination in Numerical Analysis

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9am–12 Noon
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5829 AP&M

NAME ____________________

#1.1	20	
#1.2	20	
#1.3	20	
#2.1	20	
#2.2	20	
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#3.1	20	
#3.2	20	
Total	160	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition numbers and Linear Equations

Question 1.1.

(a) Let $\Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$. Prove that for all $1 \le p \le \infty$,

$$\|\Delta\|_p = \max_{1\le i\le n} |\delta_i|.$$

(b) Let A and B be any pair of matrices such that the product AB is defined. Prove that $\|AB\|_F \le \|A\|_2 \|B\|_F$.

(c) Let $\|\cdot\|$ and $\|\cdot\|_D$ denote any vector norm and its corresponding dual norm. If $A \in \mathbb{C}^{n\times n}$, let $\|A\|_D$ denote the matrix norm subordinate to $\|\cdot\|_D$. Prove that if x, $y \in \mathbb{C}^n$ then

$$\|xy^H\| = \|x\|\,\|y\|_D.$$

Question 1.2.

(a) Consider the subtraction $x = a - b$ of two real numbers a and b such that $a \ne b$. Suppose that $\widetilde{a}$ and $\widetilde{b}$ are the result of making a *relative* perturbation Δa and Δb to a and b. Find the relative error in $\widetilde{x} = \widetilde{a} - \widetilde{b}$ as an approximation to x and hence find a condition number for the operation of subtraction. *Assume that all calculations are done in exact arithmetic.*

(b) State the *standard rounding-error model* for floating-point arithmetic. Given three representable numbers a, b and c, compute the backward and forward relative error for the floating-point value $\hat{s}$ of the calculation $s = ab+c$. Describe a situation in which $\hat{s}$ has large forward error, but small backward error.

Question 1.3.

(a) Prove that every nonsingular symmetric matrix A can be written in the form $PAP^T = LBL^T$, where P is a permutation, L is unit lower triangular and B is a block-diagonal matrix with diagonal blocks of order at most one or two.

(b) *Briefly* describe the diagonal *complete* pivoting method for finding the factorization $PAP^T = LBL^T$. Show that $\|L\|$ is bounded independently of A.

2. Least-Squares and Eigenvalues

Question 2.1. Let A be an $m \times n$ with rank r. Assume that $b \in \text{range}(A)$.

(a) Derive necessary and sufficient conditions for x to be the least-length solution of $Ax = b$ and prove that the least-length solution is unique.

(b) Define an algorithm for computing the general solution of $Ax = b$ using the QR factorization of A^T with column interchanges.

(c) Use part (b) to define the least-length solution. Verify that your algorithm gives the solution of least length.

Question 2.2. Consider a non-defective matrix $A \in \mathbb{C}^{2\times 2}$ such that

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}.$$

(a) Find the left and right eigenvectors of A.

(b) Find the condition number of each of the eigenvalues of A.

(c) Briefly discuss the situation where A is close to a defective matrix.

Question 2.3. Let $A \in \mathbb{C}^{n\times n}$. Given an approximate eigenpair (λ, u), describe how you would use one step of inverse iteration to find an improved eigenvector v of A. Hence show that (λ, v) is an exact eigenpair of $A + E$ where E may be chosen to satisfy

$$\|E\|_F = \frac{\|u\|_2}{\|v\|_2}.$$

3. Interpolation, Approximation and ODEs

Question 3.1. Consider the function $f(x) = 2x^3 - x^2 + 1$ on $[0, 2]$.

(a) Construct the (unique) quadratic interpolation polynomial $p_2(x)$ which interpolates $f(x)$ at $x = 0, 1, 2$.

(b) Derive a bound on the error $|f(x) - p_2(x)|$ which is valid over the interval $[0, 2]$.

(c) Use Simpson's rule based on $p_2(x)$ to compute an approximation to

$$\mathcal{I}(f) = \int_0^2 f(x)dx,$$

and give an expression for the error in the approximation.

(d) Derive a bound on the error in the finite difference approximation:

$$f'(x) = \left[\frac{f(x+h) - f(x-h)}{2h}\right].$$

Question 3.2. Consider the problem of best L^p-approximation of a (continuous) function $u(x)$ over the interval $[0, 1]$ from a subspace $V \subset L^p([0, 1])$: Find $u^* \in V$ such that

$$\|u - u^*\|_{L^p} = \inf_{v \in V} \|u - v\|_{L^p},$$

where

$$\|u\|_{L^p} = \left(\int_0^1 |u|^p \, dx\right)^{1/p}, \; 1 \le p < \infty, \qquad \|u\|_{L^\infty} = \sup_{x \in [0,1]} |u(x)|.$$

We wish to find the best L^p-approximation of the specific function $u(x) = x^4$.

(a) Determine the best L^2-approximation in the subspace of quadratic functions; i.e., $V = \text{span}\{1, x, x^2\}$, and justify the technique you use.

(b) Why (specifically) does this problem become tremendously more difficult if we consider the case $p \neq 2$?

(c) Prove that the decomposition of an element of a Hilbert space using the Projection Theorem is unique.