MATH 220. Complex Analysis
Qualifying exam. May 19, 2005

General instructions: 3 hours. No books or notes. Be sure to motivate all (nontrivial) claims and statements. You may use without proof any result proved in the text. You need to reprove any result given as an exercise. The notation $B(a,r) := \{z\colon |z-a| < r\}$ and $\mathbb{D} := B(0,1)$ will be used.

1. For $w \in \mathbb{D}$, consider the equation in z

(*) $$z^5(z-2) = w.$$

(a) Show that (*) has 5 roots in $\mathbb{D}$ (possibly with multiplicity) . Show that for $w \in \mathbb{D}\setminus\{0\}$, the 5 roots are distinct.

(b) Show that (*) has exactly one simple root in $B(2,1)$.

2. Let $f \in H(\mathbb{D})$, and $c_1, \ldots, c_n \in \mathbb{D}\setminus\{0\}$ with $f(c_j) = 0$ for $j = 1, \ldots, n$. Show that if $|f(z)| \le M$ for $|z| < 1$, then

$$|f(0)| \le M \prod_{j=1}^{n} |c_j|.$$

[*Hint*: Consider the function $g(z) = f(z)/\prod_j \phi_{c_j}(z)$, where ϕ_{c_j} is a one-to-one analytic map of $\mathbb{D}$ onto itself vanishing at c_j.]

3. Let $G \subset \mathbb{C}$ be a connected open set with $0 \in G$, and $f \in H(G)$, with $f(0) = 0$, $f'(0) = 1$, and $f(G) \subset G$.

(a) Show that if $G \ne \mathbb{C}$ and G is *simply connected* (not necessarily bounded) then $f(z) \equiv z$. Does the same conclusion hold for $G = \mathbb{C}$?

(b) Show that if G is *bounded* (not necessrily simply connected) then $f(z) \equiv z$.
[*Hint*: Part (a) is not used here. Prove (b) by contradiction. Consdier the the n-th iterate, $f_n := f \circ f \circ \ldots \circ f$ (n times), and compute the first non-vanishing cefficient of the Taylor series of $f_n(z) - z$ at 0 in terms of that of $f(z) - z$.]

4. Let $K \subset \mathbb{C}$ be a compact set. Assume that $\mathbb{C}\setminus K$ is connected and $0 \notin K$. Prove that for any analytic function f in an open neighborhooh of K nowhere vanishing on K, and any $\epsilon > 0$, there exists a polynomial P, satisfying

$$|f(z)P(z) - 1| \le \epsilon, \quad \forall z \in K.$$

5. Let n be a positive integer and a real number $\lambda > 1$. Consider the equation

(**) $$z^n - e^{z-\lambda} = 0$$

(a) Find all the roots of (**) with $|z| = 1$.

(b) Show that there are exactly n distinct simple roots of (**) with $|z| < 1$.

6. Show that there is an analytic function f in $\mathbb{C}\setminus[-1,1]$ such that $\big(f(z)\big)^2 = z^2 - 1$. Does f extend anlytically to $\mathbb{C}_\infty \setminus [-1,1]$? How many such functions are there?