Algebra/Applied Algebra Qualifying Exam
Part 1
May 28, 2004

(15) 1. State and prove the Schur Decomposition Theorem.

(13) 2. (a) Let $A \in M_n$. Show that if $z^H A z = 0$ for all $z \in \mathbb{C}^n$ then $A = 0$.

(b) Give a 2×2 example of $A \in M_2(\mathbb{R})$, where $A \neq 0$, but $x^T A x = 0$ for all $x \in \mathbb{R}^2$.

(c) Show that $A \in M_n$ is normal iff $\|Ax\|_2 = \|A^H x\|_2$ for all $x \in \mathbb{C}^n$.

(d) Show that $A \in M_n$ is normal iff $\theta(Ax, Ay) = \theta(A^H x, A^H y)$ for all $x, y \in \mathbb{C}^n$.

(12) 3. Let $\hat{x}$ be a least squares solution to $Ax = b$, where $A \in M_{m,n}$ and $m \geq n$. Let $A^\dagger$ be the pseudo-inverse of A. Use the Singular Value Decomposition to show that $\tilde{x} = A^\dagger b$ is the min 2- norm least squares solution to $Ax = b$, i.e., show

(a) $\tilde{x}$ is a least squares solution,

(b) if $\hat{x}$ is a least square solution then $\|\hat{x}\|_2 \geq \|\tilde{x}\|_2$, and

(c) $\tilde{x}$ is unique.

Applied Algebra Qualifying Exam, Part III: Spring 2004, May 30 2004

This part will count 60% of total points of the exam.
Do as many problems as you can but you must do at least 3 problems from 1-7 and 2 problems from 8-11.

Let $N = \{0, 1, 2, \ldots\}$, $Z = \{0, \pm 1, \pm 2, \ldots\}$, Q equal the rationals and C denote the complex numbers.

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is a partition of n, let A^λ denote the irreducible representation of the symmetric group S_n such that the Frobenius image of χ^{A^λ} is the Schur function $S_\lambda(x_1, \ldots, x_N)$ where $N > n$.

(1) (30 pts) Let G and H be a finite groups, $A : G \to GL(n, \mathrm{C})$ be a representation of G and $B : H \to GL(m, \mathrm{C})$ be a representation of H.

(a) Define the representation $A \times B : G \times H \to GL(n \cdot m, \mathrm{C})$

(b) Show that if A and B are irreducible representations, then $A \times B$ is irreducible.

(c) Show that every irreducible representation of $G \times H$ is of the form $A \times B$ where A is an irreducible representation of G and B is an irreducible representation of H.

(2) (30 pts)
(a) Let T be the trivial representation. Decompose $T \uparrow_{S_3 \times S_3}^{S_6}$ as a sum of irreducible representations of S_6 where $S_3 \times S_3$ is the Young subgroup of S_6 consisting of all permuations $\sigma \in S_6$ such that

$$\sigma(1), \sigma(2), \sigma(3) \in \{1, 2, 3\}, \sigma(4), \sigma(5), \sigma(6) \in \{4, 5, 6\}.$$

(b) Decompose $A^{(2,4)} \otimes A^{(1,5)}$ as a sum of irreducible representations of S_6 where $\otimes$ represents the Kronecker product of the representations.

(c) Decompose $A^{(1,2)} \times A^{(1,2)} \uparrow_{S_3 \times S_3}^{S_6}$ as a sum of irreducible representations of S_6. (Here $S_3 \times S_3$ is group described in part (a).)

(3) (30 pts)
(a) Use the Murnaghnam-Nakayama rule to compute the values of the character $A^{(1,4)}$ on the conjugacy classes of S_5.

(b) Express $\chi^{A^{(1,4)} \downarrow_{S_3 \times S_2}^{S_5}}$ as a sum of irreducible characters of $S_3 \times S_2$. Here $S_3 \times S_2$ is the Young subgroup of S_5 consisting of all permuations $\sigma \in S_6$ such that

$$\sigma(1), \sigma(2), \sigma(3) \in \{1, 2, 3\}, \sigma(4), \sigma(5) \in \{4, 5\}.$$

(4) (30 pts) Let G be the group of order 8 defined by the relations

$$a^4 = 1 \text{ and } a^2 = b^2 = 1 \text{ and } b^{-1}ab = a^3.$$

(a) Show that $ab = b^3a$ and that every element of G is of the form b^k or $b^k a$ where $k = 0, \ldots, 3$.

(b) Given that the conjugacy classes of G are
$C_1 = \{1\}$
$C_2 = \{b^2\}$
$C_3 = \{b, b^3\}$
$C_4 = \{a, b^2a\}$
$C_5 = \{ba, b^3a\}$
(i) Show that $H = \{1, b^2\}$ is a normal subgroup of G for which G/H is isomorphic to $Z_2 \times Z_2$.
(ii) Give the character character table for the lifting of the 4 linear characters of G/H to G.
(iii) Find the complete character table for G.

(5) (30 pts) Let H be a subgroup of G and let $G = \tau_1 H + \ldots + \tau_k H$ be its coset decomposition. Define a permutation representation L of G by

$$\begin{aligned}\sigma\langle \tau_1 H, \ldots, \tau_k H\rangle &= \langle \sigma\tau_1 H, \ldots, \sigma\tau_k H\rangle \\ &= \langle \tau_1 H, \ldots, \tau_k H\rangle L(\sigma)\end{aligned}$$

so that

$$L(\sigma)_{i,j} = \chi(\tau_i H = \sigma\tau_j H)$$

(a) Prove that L is a representation.

(b) Consider the special case where $G = S_n$ and $H = S_{n-1} \times S_1 = \{\sigma \in S_n : \sigma(n) = n\}$.

(i) Show that the coset decompostion of G relative to H is given by $G = H + (1,n)H + \ldots (n-1,n)H$ where (i,n) denotes the transposition which interchanges i and n.

(ii) Show that $\chi^L(\sigma) = fix(\sigma)$ where $fix(\sigma)$ denotes the number of fixed points of σ.

(c) In the special case where $G = S_4$ and $H = S_3 \times S_1$, use part (b) to decompose L a sum of irreducible representations of S_4.

(6) (30 pts) If $S = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ is a subset of $\{1, 2, \ldots, n\}$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a permutation let $\sigma(S)$ denote the subset $\sigma(S) = \{\sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k}\}$.. In this manner, we can define an action of S_n on the k-subsets of $\{1, 2, \ldots, n\}$ and induce a representation $A^{(k,n)}$ such that if $S_1, \ldots, S_{\binom{n}{k}})$ is a list of the k-element subsets of $\{1, \ldots, n\}$, then

$$\begin{aligned}\sigma\langle S_1, \ldots, S_{\binom{n}{k}}\rangle &= \langle \sigma(S_1), \ldots, \sigma(S_{\binom{n}{k}})\rangle \\ &= \langle S_1, \ldots, S_{\binom{n}{k}})\rangle A^{(k,n)}(\sigma).\end{aligned}$$

Let $\chi^{(k,n)}$ the character of $A^{(k,n)}$. Find the Frobenius image of $\chi^{(2,4)}$

(b) Use your result in (a) to compute the decomposition of $\chi^{(2,4)}$ into a sum of irreducible characters of S_4.

(7) (30 pts) Let H be a subgroup of a finite group G and $A : H \to GL(n, C)$ be a representation of H.

(a) Prove that for any character ϕ of G, $\langle \chi^{A\uparrow_H^G}, \phi\rangle_G = \langle \chi^A, \phi \downarrow_H^G\rangle$.

(b) Given an example to show that it is not always the case that if A is irreducible, then $A \uparrow_H^G$ is irreducible.

(c) Prove that if K is a subgroup of H and $B : K \to GL(m, C)$ is a representation of K, then the representation $B \uparrow_K^G$ is similar to the representation $(B \uparrow_K^H) \uparrow_H^G$.

(8) (30 pts) Let $\mathcal{R} = (R, +, \cdot)$ be an integral domain. Let C be the additive group of R generated by the indentity, that is, let C be the smallest subgroup of R such that C contains 0 and 1 and C is closed under +.

(a) Show that $C = \{n \cdot 1 : n \in Z\}$ and hence is a subring of R. Here if $n \geq 0$, then we can define $n \cdot 1$ by induction as $0 \cdot 1 = 0$, $1 \cdot 1 = 1$ and $(n+1) \cdot 1 = 1 + (n \cdot 1)$ and if $n < 0$, we define $n \cdot 1 = -(|n| \cdot 1)$.

Show that $\phi : Z \to C$ defined by $\phi(n) = n \cdot 1$ is a surjective ring homomorphism.

(c) Prove that either C is isomorphic to Z or C is isomorphic to Z_p for some prime p.

(9) (40 pts.)

Consider the equations

$$\begin{aligned} x^2 + 2y^2 &= 2 \\ x^2 - xy + y^2 &= 1 \end{aligned}$$

(a) Let I be the ideal of $\mathbf{C}[x,y]$ generated by these equations. Find the Groebner basis for I relative to lexicographic order where $y > x$.

(b) Find a Groebner basis for $\mathbf{C}[x] \cap I$.

(c) Find all solutions to these equations that lie $\mathbf{C}^2$.

(d) Find a vector space basis for $\mathbf{C}[x,y]/I$.

(10) (40 pts) Let I and J be an ideals in $k[x_1, \ldots, x_n]$ where k is field.

(a) Show that $\sqrt{\sqrt{I}} = \sqrt{I}$.

(b) Show that $\sqrt{I \cap J} = \sqrt{IJ} = \sqrt{I} \cap \sqrt{J}$

(c) Is $x^2 - y^2 \in \sqrt{\langle x^2 + x, x^2 - y \rangle}$?

(d) Is $x^2 + y^2 \in \sqrt{\langle x + y, x^2 - y \rangle}$?

(11) (40 pts) Let

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that A generates a cyclic group G of order 4.

(b) Show that if $f \in \mathbf{C}[x,y]^G$, then f can have no monomials of odd degree.

(b) Use Molien's Theorem to show that the Hilbert series of G is

$$\phi_G(z) = \frac{1 + z^4}{(1 - z^2)(1 - z^4)}.$$

(c) Show that $\mathbf{C}[x,y]^G$ is Cohen-Macauly by explicitly finding the generators and separators for $\mathbf{C}[x,y]^G$.