Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiners: Li-Tien Cheng and Philip E. Gill

1:00–4:00pm
Thursday May 29, 2003
7421 AP&M

NAME ____________________

#1.1	20	
#1.2	20	
#1.3	20	
#2.1	20	
#2.2	20	
#2.3	20	
#3.1	20	
#3.2	20	
Total	160	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition numbers and Linear Equations

Question 1.1. Assume that $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times k}$.

(a) Define the one-norm $\|A\|_1$, two-norm $\|A\|_2$, infinity norm $\|A\|_\infty$, and Frobenius norm $\|A\|_F$ of A. Prove that $\|A\|_2 = \sigma_1$, where σ_1 is the largest singular value of A.

(b) Establish the following bounds and identities:

- **(i)** $\|A^H\|_2 = \|A\|_2$.
- **(ii)** $\|A\|_2^2 \le \|A\|_1 \, \|A\|_\infty$.
- **(iii)** $\| \, |A| \, \|_2 \le \sqrt{\text{rank}(A)}\|A\|_2$.
- **(iv)** $\|AB\|_F \le \|A\|_F\|B\|_2$ and $\|AB\|_F \le \|A\|_2\|B\|_F$.

Question 1.2.

(a) Prove that the vector two-norm is self-dual.

(b) Let A be a nonsingular matrix of order n. Prove that

$$\frac{1}{\text{cond}(A)} = \min_{E\in\mathbb{R}^{n\times n}} \{\|E\|_2/\|A\|_2 \mid A+E \;\; \text{singular}\},$$

where cond(A) denotes the spectral condition number of A. Comment on the uniqueness of E.

(c) Given the matrix

$$A = \begin{pmatrix} \frac{5}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ \frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} \end{pmatrix},$$

find the matrix E that solves $\min_{E\in\mathbb{R}^{2\times 2}}\{\|E\|_2/\|A\|_2 \mid A+E \;\; \text{singular}\}$,

Question 1.3.

(a) State the *standard rounding-error model* for floating-point arithmetic.

(b) Let $\mathbf{u}$ denote the unit roundoff, and assume that $n\mathbf{u} < 1$ for the positive integer n. If $\{\delta_i\}$ are n scalars such that $|\delta_i| \le \mathbf{u}$, prove that

$$\prod_{i=1}^{n}(1+\delta_i) = 1 + \theta_n, \quad \text{where} \quad |\theta_n| \le \gamma_n,$$

with $\gamma_n = n\mathbf{u}/(1-n\mathbf{u})$.

(c) Let A and B be square matrices of order n. Consider the matrix product $C = AB$.

- **(i)** If the computed value of C is written as $\widehat{C} = C + E$, derive the bound $\|E\|_2 \le \gamma_n\sqrt{n}\|A\|_2 \, \|B\|_2$.
- **(ii)** Hence prove that multiplication by an orthogonal matrix is backward stable.

2. Least-Squares and Eigenvalues

Question 2.1. Let A be a real $m \times n$ matrix with $m < n$. Assume $\text{rank}(A) = m$. We want to solve the homogeneous system of linear equations $Ax = 0$.

(a) How would you implement Gaussian elimination to solve this system for a non-trivial solution?

(b) How would you use a QR decomposition to obtain a non-trivial x?

(c) How would you use an SVD to obtain a non-trivial x?

(d) If $Ax = 0$ and $x \neq 0$, determine a second solution y such that $Ay = 0$, $y \neq 0$ and $x^T y = 0$, using one of the above methods.

Question 2.2. Consider a matrix $A \in \mathbb{C}^{n\times n}$ with Schur decomposition $Q^H A Q = T$. Assume that T can be partitioned as

$$T = \begin{pmatrix} T_{11} & t_{12} & T_{13} \\ & \lambda & t_{23}^H \\ & & T_{33} \end{pmatrix},$$

where T_{11} and T_{33} are square and upper triangular, and λ does not occur on the diagonal of either T_{11} or T_{33}. Find the condition number of the eigenvalue λ.

Question 2.3. Let $A \in \mathbb{C}^{n\times n}$.

(a) Let (λ, v) be an approximate eigenpair of A such that $\|v\|_2 = 1$ and $\lambda \notin \lambda(A)$. Show that there exists a matrix E with $\|E\|_F = \|Av - \lambda v\|_2$ such that (λ, v) is an exact eigenpair of $A + E$.

(b) Let $u \in \mathbb{C}^n$ and $\lambda \notin \lambda(A)$ be given. Show that the vector $v = (A - \lambda I)^{-1} u$ is an eigenvector of $A + E$, where E may be chosen to satisfy

$$\|E\|_F = \frac{\|u\|_2}{\|v\|_2}.$$

(c) Use the result of part (b) to comment on the effectiveness of finding an approximate eigenvector by inverse iteration

3. Interpolation, Approximation and ODEs

Question 3.1. Let $f(x)$ be a smooth function and $h > 0$ a constant.

(a) Let $P(x)$ be the cubic Hermite interpolatory polynomial satisfying

$$P(0) = f(0), P(h) = f(h), P'(0) = f'(0), P'(h) = f'(h).$$

Construct $P(x)$ and show

$$P\left(\frac{h}{2}\right) = \frac{1}{2}[f(0) + f(h)] + \frac{h}{8}[f'(0) - f'(h)].$$

(b) Use Simpson's Rule on the integral of f over $[0, h]$, the expression for $P\left(\frac{h}{2}\right)$, and the fact that P approximates f to derive the corrected Trapezoidal rule

$$\int_0^h f(x)\, dx = \frac{h}{2}[f(0) + f(h)] + \frac{h^2}{12}[f'(0) - f'(h)] + E(x, h).$$

Write down an expression for the error term E.

Question 3.2.

Let $f(x) = e^{\sin x}$ and consider the interval $[0, 8]$.

(a) Derive the equation for the interpolatory polynomial of $f(x)$ that uses Chebyshev polynomials to determine the optimal locations for $n + 1$ nodes in $[0, 8]$. You do not need to simplify the form of the polynomial. In what sense are the node locations optimal?

(b) Let $P(x)$ be the piecewise linear interpolant of $f(x)$ using the uniformly spaced nodes $0 = x_0 < x_1 < \cdots < x_n = 8$ with stepsize h. Estimate from error bounds the n needed such that

$$\max_{0 \le x \le 8} |f(x) - P(x)| < 10^{-6}$$

will be satisfied. You do not need to simplify arithmetic in the result.

(c) Consider composite Trapezoidal rule using nodes at $0 = x_0 < x_1 < \cdots < x_n = 8$. Call the resulting approximation A_n. Estimate from error bounds the n needed such that

$$\left| \int_0^8 f(x)\, dx - A_n \right| < 10^{-6}$$

will be satisfied. You do not need to simplify arithmetic in the result.