Numerical Analysis Qualifying Examination

May 31, 2002

NAME ____________________

SIGNATURE ____________________

#1	15	
#2	15	
#3	10	
Total	40	

Question 1. Let $f^* \in \mathcal{S}$ be the continuous piecewise linear interpolant for f on a mesh of $n+1$ knots $x_0 < x_1 < \ldots < x_n$. Let $h = \max_i(x_i - x_{i-1})$ and assume $f \in C^2(x_0, x_n)$. Prove

$$\| f - f^* \|_\infty \leq \frac{h^2}{8} \| f'' \|_\infty$$

Question 2. Let

$$\mathcal{I}(f) = \int_{-1}^{1} f(x)dx$$

Consider the two point Gauss-Legendre quadrature formula of the form

$$\mathcal{Q}(f) = w_1 f(x_1) + w_2 f(x_2) \tag{1}$$

a. Find the knots x_1 and x_2 and the weights w_1 and w_2 for the Gauss-Legendre formula (1).

b. Derive an error estimate for $\mathcal{E}(f) = |\mathcal{I}(f) - \mathcal{Q}(f)|$. Be sure to explicitly evaluate the constant.

Question 3. Prove Gronwall's Lemma: Let

$$y' \leq \kappa y + \tau$$

for $0 \leq t \leq T$, and $\tau, \kappa, y \geq 0$, τ and κ constant. Then

$$\max_{0 \leq t \leq T} y(t) \leq e^{\kappa T} y(0) + \frac{\tau}{\kappa}\left(e^{\kappa T} - 1\right).$$

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# 1	15	
# 2	15	
# 3	15	
# 4	20	
# 5	20	
# 6	25	
Part AB	110	
Part C	40	
Total	150	

(15) 1. State and prove the SVD Existence Theorem (for real $m \times n$ matrices).

(15) 2. Let A be $m \times n, \text{rank}(A) = r$. Use the SVD to prove:

(a) $\|A\|_2 \leq \|A\|_F \leq \sqrt{r}\ \|A\|_2$

(b) $\text{rank}(A) = \text{rank}(A^T A) = \text{rank}(AA^T)$

(c) $\sigma_n(A)^2 x^T x \leq x^T(A^T A)x \leq \sigma_1(A)^2 x^T x$ for all $x \in \mathbb{R}^n, m \geq n$.

(15) 3. (a) Let D be an $m \times n$ diagonal matrix. Prove $\|D\|_p = \max_i |d_{ii}|$ for $1 \leq p \leq \infty$.

(b) Prove that if A is $m \times n, \text{rank}(A) = n$ and $\|E\|_p\ \|A^+\|_p < 1$ for some $p, 1 \leq p \leq \infty$, then $\text{rank}(A + E) = n$.

(c) Let A be $n \times n$, nonsingular, and $A = QR$, where Q is orthogonal and R is upper triangular with positive diagonal. Prove that Q and R are unique.

(20) 4. Let the computed L and U satisfy $A + E = LU$, where L is unit lower triangular and U is upper triangular. Derive the bound on E:

$$|E_{ij}| \leq (3 + u)u \max(i - 1, j)g,$$

where $g = \max_k \max_{i,j} |a_{ij}^{(k)}|$ and $u =$ machine precision.

(20) 5. (a) Prove that $\hat{x}$ is a least squares solution for $r(x) = Ax - b$, where A is $m \times n, m \geq n$, iff $\hat{x}$ satisfies the normal equations.

(b) Let $Ax = b$, where A is $m \times n, m < n, \text{rank}(A) = r = \text{rank}[A|b]$. Derive the min 2-norm solution to $Ax = b$ in terms of the SVD of A.

(5) 6. (a) Show $A, n \times n$, has n linearly independent eigenvectors iff A is diagonalizable.

(10) (b) Let $r = Ax - \lambda x, \|x\|_2 = 1$. Find E such that $(A+E)x = \lambda x$ and $\|E\|_2 = \|r\|_2$, where A, E are $n \times n$, complex.

(10) (c) Show when and how the generalized symmetric eigenproblem, $Ax = \lambda Bx, x \neq 0, A = A^T, B = B^T$, can be reduced to a standard symmetric eigenproblem, $My = \mu y, y \neq 0, M = M^T$.