Topology Qualifying Exam, Spring 2000

1. Let X be a path connected topological space which is NOT compact. Show that $H^0_{comp}(X) = 0$.

2. Compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Q}/\mathbb{Z})$.

3. Compute the cohomology ring of $\mathbb{CP}^3 \times \mathbb{RP}^2$.

4. Let X be the suspension of $\mathbb{RP}^2$, i.e.
$$X = \mathbb{RP}^2 \times [0,1]/(\mathbb{RP}^2 \times \{0\}), (\mathbb{RP}^2 \times \{1\}).$$
Prove that X is not homotopy equivalent to a compact manifold.

For the last two problems we'll need the following **definition**: If M, N are connected closed oriented manifolds of the same dimension and $f : M \to N$ is a continuous map, then the *degree* $\deg(f) \in \mathbb{Z}$ is defined by
$$f_*[M] = \deg(f) \cdot [N] \in H_n(N;\mathbb{Z}).$$

5. Prove that any continuous map $f : S^2 \times S^2 \to \mathbb{CP}^2$ has even degree.

6. Let M^n be a connected closed oriented manifold and assume that there is a degree one map $f : S^n \to M^n$. Show that $H_i(M;\mathbb{F}) = 0$ for all $0 < i < n$ and any field $\mathbb{F}$.