Math 281 **Qualifying Exam.** **June 15, 2000**

Note: In what follows, all p.d.f.'s are densities with respect to Lebesgue measure on the real line.

(1) Assume X_1, X_2 is a random sample of size 2 from the distribution having p.d.f.

$$f(x) = e^{-x}, \qquad 0 < x < \infty$$
$$= 0 \quad \text{elsewhere} .$$

a. Find the p.d.f. of $W = aX_1 + bX_2$ for any constants a and b. Then compute the mean and variance of this distribution.

b. Find the p.d.f. of $Y = \max\{ X_1, X_2 \}$.

(2) Suppose that X has a Poisson distribution with parameter λ . Now let the parameter $\theta = e^{-2\lambda}$ be estimated by $\hat{\theta} = (-1)^X$.

a. Show that this estimator is unbiased.

b. Further, show that it is UMVUE.

But this estimator is absurd, why ? Comment on the theory of UMVUE in light of this example.

(3) Suppose the p.d.f. of X is given by f(x) as stated in problem (1). Compute the characteristic function and then

$$\mu = E(X) \quad \text{and} \quad \sigma^2 = Var(X).$$

Now suppose a random sample of size n is taken from this distribution where the sample mean is given by $\overline{X}_n$. Use a characteristic function argument to prove that

$$\sqrt{n}\ (\overline{X}_n - \mu)/\sigma \xrightarrow{L} Z \sim N(0,1)\ .$$

(4) Assume $X_1, X_2, \ldots, X_n$ is a random sample taken from a distribution whose p.d.f. is given by

$$f(x;\theta) = \frac{1}{2\theta^3} x^2 e^{-x/\theta}; \quad 0 < \theta < \infty, \quad 0 < x < \infty$$

$$= 0 \quad \text{elsewhere .}$$

Find the Cramer-Rao lower bound and thus the asymptotic variance of the maximum likelihood estimator of θ.

(5) Find the distribution of the range of a random sample of size n where the p.d.f. is given by

$$f(x;\theta) = 4e^{-4x}; \quad 0 < x < \infty$$
$$= 0 \quad \text{elsewhere}.$$

(6) Let $X_{(1)} < X_{(2)} < \ldots < X_{(n)}$ be the order statistics for a random sample (of size n) from the distribution given by Problem (1).

a. Show that $X_{(r)}$ and $X_{(s)} - X_{(r)}$ are independent for s>r.

b. Find the distribution of $X_{(r+1)} - X_{(r)}$.

c. Interpret the significance of these results if the sample arose from a life test on n light bulbs with exponential lifetimes (i.e., the distribution from Problem (1)).