

Ph.D./Masters Qualifying Examination
in Numerical Analysis

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10am-1pm
Wednesday May 30, 2007
5402 AP&M

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- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition Numbers, Linear Equations and Linear Least-Squares

In Parts 1 and 2, $\|\cdot\|_p$ refers to the vector p -norm or its subordinate matrix norm.

Question 1.1.

- (a) Given any $x \in \mathbb{C}^m$, find positive constants c_1 and c_2 , independent of x such that

$$c_1 \|x\|_2 \leq \|x\|_\infty \leq c_2 \|x\|_2.$$

- (b) If $A \in \mathbb{C}^{m \times n}$, prove that $\|A\|_2 = \sigma_1$, where σ_1 is the largest singular value of A .
- (c) Assume that $A \in \mathbb{C}^{m \times n}$ has rank r . Find a scalar σ ($\sigma > 0$), independent of p , such that

$$\|Ap\|_\infty \geq \sigma \|p\|_2 \quad \text{for all } p \in \text{range}(A^T).$$

Question 1.2.

- (a) State the *standard rounding-error model* for floating-point arithmetic.
- (b) Let u denote the unit round-off. Let n be a positive integer such that $nu < 1$. If $\{\delta_i\}$ is a set of n numbers such that $|\delta_i| \leq u$, and $\{s_i\}$ are integers such that $s_i = \pm 1$, prove that

$$\prod_{i=1}^n (1 + \delta_i)^{s_i} = 1 + \theta_n,$$

where $|\theta_n| \leq \gamma_n$, with $\gamma_n = nu/(1 - nu)$.

- (c) Given two n -vectors x and y , let \widehat{Z} denote the *computed* version of the rank-one matrix $Z = xy^T$. Apply the standard rounding error model to derive a bound for the component-wise forward error in \widehat{Z} as an approximation to Z . Is the calculation of \widehat{Z} backward stable? Justify your answer.

Question 1.3. Assume that $A \in \mathbb{R}^{n \times n}$.

- (a) Suppose that r ($r < n$) steps of Householder reduction with column interchanges gives the decomposition

$$AP = Q \begin{pmatrix} R_{11} & R_{12} \\ & 0 \end{pmatrix},$$

where Q is orthogonal, P is a permutation and R_{11} is an $r \times r$ nonsingular upper triangle. Define bases for $\text{null}(A)$ and $\text{range}(A^T)$ in terms of the QR factors above. Verify that the proposed bases satisfy the properties of a basis.

- (b) Now assume that r steps of Householder reduction give:

$$AP = Q \begin{pmatrix} R_{11} & R_{12} \\ & E \end{pmatrix},$$

where Q is orthogonal, P is a permutation and R_{11} is an $r \times r$ nonsingular upper triangle. Show that σ_n , the smallest singular value of A , satisfies $\sigma_n \leq \|E\|_2$. Give a *brief* discussion of the implication of this result.

2. Nonlinear Equations, Nonlinear Least-Squares and Optimization

Question 2.1.

- (a) Let $F : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ be continuously differentiable on the open convex set \mathcal{D} . Compute the Fréchet derivative for the function $f : \mathbb{R}^n \mapsto \mathbb{R}$ such that $f(x) = \|x\|_2$.
- (b) Given a real $n \times n$ symmetric matrix A , find the Fréchet derivative of the function $G : \mathbb{R}^{n+1} \mapsto \mathbb{R}^{n+1}$ such that

$$G(x, \lambda) = \begin{pmatrix} Ax - \lambda x \\ \|x\|_2 - 1 \end{pmatrix}.$$

Hence define an iteration of Newton's method for finding an eigenvalue of A and its associated eigenvector.

- (c) An eigenvalue of a matrix is *simple* if it has algebraic multiplicity 1. If λ^* is a simple eigenvalue of A and x^* is its corresponding normalized eigenvector, prove that $G'(x^*, \lambda^*)$ is nonsingular. Give a *brief* discussion of the implication of this result when finding x^* and λ^* using Newton's method.

Question 2.2. Consider the function $f : \mathbb{R}^3 \mapsto \mathbb{R}$ such that

$$f(x) = x_1^2 + x_2^2 \cos x_3 - e^{x_2} x_3^2 + 4x_3.$$

- (a) Compute the spectral decomposition of the Hessian matrix of second derivatives at $\bar{x} = (0, 1, 0)^T$.
- (b) Compute the Newton direction p^N and modified Newton direction p^M at \bar{x} . Determine if p^N and p^M are descent directions.
- (c) Find a direction of negative curvature that is a direction of decrease for f at \bar{x} .

Question 2.3.

- (a) Find all the eigenvalues of the matrix $I + \gamma uv^T$, where γ is a scalar and u and v are n vectors.
- (b) Given an $n \times n$ symmetric positive-definite matrix B , and n -vectors y and s , consider the symmetric rank-one quasi-Newton update

$$B_+ = B + \frac{1}{(y - Bs)^T s} (y - Bs)(y - Bs)^T. \quad (2.1)$$

- (i) Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a quadratic function with a symmetric positive-definite Hessian matrix. Let $s = x_+ - x$ and $y = \nabla f(x_+) - \nabla f(x)$, where $\nabla f(x)$ is the gradient of f evaluated at x . If vectors $\bar{s} = \bar{x}_+ - \bar{x}$ and $\bar{y} = \nabla f(\bar{x}_+) - \nabla f(\bar{x})$ satisfy $B\bar{s} = \bar{y}$, prove that $B_+\bar{s} = \bar{y}$.
- (ii) Find a condition on the vectors y and s that will guarantee the positive definiteness of B_+ .

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3. Approximation and Numerical ODEs

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We also denote by \mathcal{P}_n the set of all polynomials of degree $\leq n$ for any integer $n \geq 0$.

Question 3.1.

(a) Prove for any $f \in C[a, b]$ that

$$\lim_{n \rightarrow \infty} \inf_{q_n \in \mathcal{P}_n} \max_{a \leq x \leq b} |f(x) - q_n(x)| = 0,$$

$$\lim_{n \rightarrow \infty} \inf_{q_n \in \mathcal{P}_n} \int_a^b [f(x) - q_n(x)]^2 dx = 0.$$

- (b) Let $p_2 \in \mathcal{P}_2$ be the best uniform approximation in \mathcal{P}_2 of the function $g(x) = x^3 - 2x^2 + 1$ with respect to the $C[-1, 1]$ -norm. What is the value of $p_2(1)$? Justify your answer.
- (c) Let Q_0, \dots, Q_n, \dots be orthogonal polynomials in $L^2[a, b]$. Fix $n \geq 1$. Prove that Q_n has n simple roots in $[a, b]$.

Question 3.2.

(a) Find the degree of precision of the numerical quadrature

$$\int_a^b f(x) dx \approx \frac{1}{2}(b-a)[f(a) + f(b)] - \frac{1}{12}(b-a)^2[f'(b) - f'(a)] \quad \forall f \in C^1[a, b].$$

(b) Consider a sequence of interpolatory numerical integration formulas

$$\int_a^b f(x) dx \approx \sum_{k=1}^n A_k^{(n)} f(x_k^{(n)}), \quad n = 1, \dots$$

Suppose all the coefficients $A_k^{(n)}$ ($k = 1, \dots, n; n = 1, \dots$) are positive. Prove that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n A_k^{(n)} f(x_k^{(n)}) = \int_a^b f(x) dx \quad \forall f \in C[a, b].$$