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Math 281 A,B Qualifying Exam. Sept. 20, 1999

(1) Assume X_1, X_2 is a random sample of size 2 from the distribution having p.d.f.

$$f(x) = e^{-x}, \quad 0 < x < \infty$$
$$= 0 \quad \text{elsewhere}.$$

a. Find the p.d.f. of $W = aX_1 + bX_2$ for any constants a and b. Then compute the mean and variance of this distribution.

b. Find the p.d.f. of $Y = \max\{ X_1, X_2 \}$.

(2) Suppose that X has a Poisson distribution with parameter λ. Now let the parameter $\theta = e^{-2\lambda}$ be estimated by $\hat{\theta} = (-1)^X$.

a. Show that this estimator is unbiased.

b. Further, show that it is UMVUE.

But this estimator is absurd, why ? Comment on the theory of UMVUE in light of this example.

(3) Let $Y \sim N_n(X\beta, \sigma^2 V)$ in which X is nxp (with n>p), rank(X) $\le$ p, β is px1, and V>0 is nxn. Assume, also, that β and σ^2 are unknown. Derive the complete, sufficient statistic for (β, σ^2). Write down the *usual* unbiased estimators for β and σ^2. Show that these estimators are indeed unbiased. Finally, obtain *general* condions under which a quadratic form in Y is an unbiased estimator of σ^2.

(4) State the Helly-Bray theorem and use this result to prove that

$$X_n \xrightarrow{L} X \;\Rightarrow\; \varphi_n(t) \to \varphi(t)$$

where X_n and X are real valued random variables with respective characteristic functions $\varphi_n(t)$ and $\varphi(t)$. (In what follows, you will need to assume that the converse is also true.)

(5) Suppose the p.d.f. of X is given by f(x) as stated in problem (1). Compute the characteristic function and then

$$\mu = E(X) \quad \text{and} \quad \sigma^2 = Var(X).$$

Now suppose a random sample of size n is taken from this distribution where the sample mean is given by $\overline{X}_n$. Use a characteristic function argument to prove that

$$\sqrt{n}\,(\overline{X}_n - \mu)/\sigma \xrightarrow{L} Z \sim N(0,1) .$$