Applied Algebra Qualifying Exam: Part II
September 6, 2011

Do as many problems as you can, but you must attempt at least 5 problems where two of the problems are from problems 1-5 and two of the problems are from problems 6-9. The point values are relative values for this part of the exam. Your final score will be scaled so that this part of the exam will represent 60% of your point total.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Q}$ equal the rationals and $\mathbb{C}$ denote the complex numbers. Suppose that $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is a partition of n. Then A^λ denotes the irreducible representation of the symmetric group S_n such that the Frobenius image of $\chi^{A^\lambda} = \chi^\lambda$ is the Schur function $S_\lambda(x_1, \ldots, x_N)$ where $N > n$ and $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ denotes the Young subgroup of S_n corresponding to λ.

(1) (30 pts)
(a) Use the Murnaghnam-Nakayama rule to compute the character table of S_4.

(b) Express $\chi^{A^{(2,2)} \downarrow^{S_4}_{S_2 \times S_2}}$ as a sum of irreducible characters of $S_2 \times S_2$. (Hint: First write out the character table for $S_2 \times S_2$.)

(2) (40 pts) Let G be the group of order 21 defined by the relations

$$a^7 = b^3 = 1 \text{ and } b^{-1}ab = a^2.$$

(a) Verify that the conjugacy classes of G are
$C_1 = \{1\}$
$C_2 = \{a, a^2, a^4\}$
$C_3\{a^3, a^5, a^6\}$
$C_4 = \{a^k b : k = 0, \ldots, 6\}$
$C_5\{a^k b^2 : k = 0, \ldots, 6\}$
(b) Show that $H = \{a^k : k = 0, \ldots, 6\}$ is a normal subgroup of G for which G/H is isomorphic to Z_3. Give the character character table for the lifting of the 3 linear characters of G/H to G.
(c) Let χ be the linear character of H given by

$$\chi(a^k) = \eta^k \text{ for } \ k = 0, \ldots 6$$

where $\eta = e^{2\pi i/7}$. Show that $\chi \uparrow_H^G$ is an irreducible character of G.
(d) Use parts (b) and (c) to give a complete character table for G.

(3) (30 pts)
(a) Find the decomposition of $A^{(2,1^2)} \times A^{(2,2)} \uparrow^{S_8}_{S_4 \times S_4}$ as a sum of irreducible representations of S_8.
(b) Let T denote the trivial representation on the Young subgroup $S_3 \times S_2 \times S_1$ of S_7 and Alt denote the alternating representation on the Young subgroup $S_3 \times S_2 \times S_1$ of S_7. Find the decompositon of

$$T \uparrow^{S_6}_{S_3 \times S_2 \times S_1} \text{ and } Alt \uparrow^{S_6}_{S_3 \times S_2 \times S_1}.$$

as a sum of irreducible representations of S_7.

(c) Find the decomposition of the Kronecker product $A^{(3,2)} \otimes A^{(3,2)}$ as a sum of irreducible representations of S_5.

(4) (30 pts) Let H be a subgroup of G and let $G = \tau_1 H + \ldots + \tau_k H$ be its coset decomposition. Define a permutation representation L of G by

$$\begin{aligned} \sigma\langle \tau_1 H, \ldots, \tau_k H\rangle &= \langle \sigma\tau_1 H, \ldots, \sigma\tau_k H\rangle && (1) \\ &= \langle \tau_1 H, \ldots, \tau_k H\rangle L(\sigma) && (2) \end{aligned}$$

so that $L(\sigma)_{i,j} = \chi(\tau_i H = \sigma\tau_j H)$.

(a) Prove that L is a representation.

(b) Consider the special case where $G = S_n$ and $H = S_{n-1} \times S_1 = \{\sigma \in S_n : \sigma(n) = n\}$.

(i) Show that the coset decompostion of G relative to H is given by $G = H + (1, n)H + \ldots (n-1, n)H$ where (i, n) denotes the transposition which interchanges i and n.

(ii) Show that $\chi^L(\sigma) = fix(\sigma)$ where $fix(\sigma)$ denotes the number of fixed points of σ.

(c) In the special case where $G = S_4$ and $H = S_3 \times S_1$, use part (b) to decompose L a sum of irreducible representations of S_4.

(5) (30 pts.) Let G and H be finite groups and let $A : G \rightarrow GL_n(C)$ and $B :\rightarrow GL_m(C)$ be representations of G and H respectively.

a) Show that $A \times B : G \times H \rightarrow GL_{nm}(C)$ is representation where for $(\sigma, \tau) \in G \times H$,

$$A \times B((\sigma, \tau)) = A(\sigma) \otimes B(\tau)$$

and for matrices M and N, $M \otimes N$ is the Kronecker product of M and N.

b) Show that if A is an irreducible representation of G and B is an irreducible representation of H, then $A \times B$ is an irreducible representation of $G \times H$.

c) Show that every irreducible representation of $G \times H$ is of the form $A \times B$ where A is an irreducible representation of G and B is an irreducible representation of H.

(6) (30 pts.) Let $\langle A, +, \cdot\rangle$ be a commutative ring with identity 1 and let $<$ be a linear order on A such that for all a, b, x in A

(I) $a < b \Rightarrow a + x < b + x$ and

(II) $a < b, 0 < x \Rightarrow a \cdot x < b \cdot x$.

(a) Prove that $\langle A, +, \cdot\rangle$ is an integral domain.

(b) Let $A^+ = \{a \in A : 0 < a\}$. Prove the following:

(i) A^+ is closed under multiplication and addition.

(ii) If $a \in A$, then exactly one of the following holds: $a \in A^+, -a \in A^+, a = 0$.

(iii) $1 \in A^+$.

(7) (40 pts.) Consider the equations

$$\begin{aligned} x^2 + y &= -2 \\ 2xy &= y^2 - 2y \end{aligned}$$

(a) Let I be the ideal of $\mathbf{C}[x,y]$ generated by these equations. Find the Groebner basis for I relative to lexicographic order where $x > y$.

(b) Find a Groebner basis for $\mathbf{C}[y] \cap I$.

(c) Find all solutions to these equations that lie $\mathbf{C}^2$.

(d) Find a vector space basis for $\mathbf{C}[x,y]/I$.

(8) (40 pts.) Let I and J be ideals in $k[x_1, \ldots, x_n]$ where k is field.

(i) Prove $I \cap J = (tI + (1-1)J) \cap k[x_1, \ldots, x_n]$.

(ii) Prove that $\mathbf{V}(\mathbf{I} \cap \mathbf{J}) = \mathbf{V}(\mathbf{I}) \cup \mathbf{V}(\mathbf{J})$ where for any set $X \subseteq k[x_1, \ldots, x_n]$, $\mathbf{V}(X)$ is the affine variety defined by X.

(iii) Prove that $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

(iv) Let $I = \langle x^3 y \rangle$ and $J = \langle xy^3 + xy \rangle$ be ideals in $k[x,y]$. Find a Gröbner basis for $I \cap J$ relative to lexicographic order where $x > y$.

(9) (30 pts.) Let k be a field.

(a) State Hilbert's Nullstellensatz Theorem.

(b) Prove that if $I = \langle f_1, \ldots, f_s \rangle \subseteq k[x_1, \ldots, x_n]$ is an ideal, then $f \in \sqrt{I}$ if and only if $1 \in \langle f_1, \ldots, f_s, 1 - yf \rangle \subseteq k[x_1, \ldots, x_n, y]$.

(c) Prove that if $f \in k[x_1, \ldots, x_n]$ and $J = \langle f \rangle$ is the principal ideal generated by f, then $\sqrt{J} = \langle f_1 f_2 \cdots f_r \rangle$ where $f = f_1^{a_1} f_2^{a_2} \cdots f_r^{a_r}$ is the factorization of f into a product of distinct irreducible polynomials in $k[x_1, \ldots, x_n]$.