2010 Fall Topology Qual

Three hour exam. Each question is worth five marks. You can use any standard facts or theorems in your work provided you state clearly that you are doing so. Please try to write good clear mathematics!

1. Let X be a space formed by gluing two distinct copies of the solid torus $S^1 \times B^2$ along their boundary $S^1 \times S^1$s, via the identity map. Calculate the fundamental group and homology groups of X.

2. How many distinct double covers does the Klein bottle have? Can you identify any of them? (Recall that the Klein bottle can be formed from a square $I \times I$ by identifying opposite edges, one pair in the parallel and one pair in the opposite direction.)

3. Show that a closed, compact, simply-connected 3-manifold M^3 is homotopy equivalent to S^3.

4. Let X be a space whose integral homology groups are $\mathbb{Z}, 0, \mathbb{Z}_8$ in dimensions $0, 1, 2$, and zero otherwise. Compute the integral homology groups of $X \times \mathbb{R}P^3$.

5. Show that there does not exist a map of degree 1 from $S^2 \times S^2$ to $\mathbb{C}P^2$.

6. An orientable closed compact 4-manifold W^4 has a finite fundamental group with d elements, and the rank of its second homology group is r. What is the rank of the second homology group of its universal cover?

7. Use the Hurewicz theorem to calculate π_2 of the space $\mathbb{R}P^2 \vee S^2 \vee S^2$ (that is, the one-point union of a projective plane and two spheres).

8. The infinite hexagonal lattice forms a covering space of the theta graph, as shown below. What is the group of deck translations (covering automorphisms) of the covering?