ALGEBRA QUALIFYING EXAM

September, 2010

Do 8 Problems

(1) Show that a finitely generated subgroup of the additive group of the rationals is cyclic.

(2) Show that a group of order $2010 = 2 * 3 * 5 * 67$ is solvable.

(3) Show that if H is a cyclic normal subgroup of G, then every subgroup of H is normal in G..

(4) Let E be a finite separable extension of F. Show that, then $E = F(a)$, for some a in E. (Hint: Use the Fundamental Theorem of Galois Theory)

(5) Let E be a finite dimensional Galois extension of a field F and let $G = Gal(E/F)$. Suppose that $G$is an abelian group. Prove that if K is any field between E and F, then K is a Galois extension of F. What is the Galois group of K over F?

(6) Explicitly determine the spitting fields over the rationals of the following two polynomials and their degrees over Q:
(a) $x^6 + 1$ and
(b) $x^6 - 1$

(7) Let R be a commutative ring with identity and let U be maximal among non-finitely generated ideals of R. Prove U is a prime ideal.

(8) Let R be a ring with identity such that the identity map is the only ring automorphism of R. Prove that the set N of all nilpotent elements of R is an ideal of R. (Hint: $1 + n$, with n a nilpotent element, is invertible.)

(9) Give an example of a right noetherian ring that is not left noetherian and an example of a module that satisfies the descending chain condition on submodules, but not the ascending chain condition on submodules.