Ph.D./Masters Qualifying Examination in Numerical Analysis

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10am to 1pm
Tuesday September 8, 2009
2402 AP&M

NAME ______________________

#1.1	30	
#1.2	30	
#1.3	30	
#2.1	30	
#2.2	30	
#2.3	30	
#3.1	30	
#3.2	30	
Total	240	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.
- Write your answers to the questions in Section 3 on separate sheets so that they may be graded separately.

1. Norms, Condition Numbers and Linear Equations

Question 1.1.

(a) Consider the subtraction $x = a - b$ of two real numbers a and b such that $a \neq b$. Suppose that $\tilde{a}$ and $\tilde{b}$ are the result of making a *relative* perturbation Δa and Δb to a and b. Find the relative error of $\tilde{x} = \tilde{a} - \tilde{b}$ as an approximation to x and hence find a condition number for the operation of subtraction. *Assume that all calculations are done in exact arithmetic.*

(b) State the *standard rounding-error model* for floating-point arithmetic. Given three representable numbers a, b and c, compute the backward and forward relative error for the floating-point value $\hat{s}$ of the expression $s = ab + c$. Describe a situation in which $\hat{s}$ has large forward error, but small backward error.

Question 1.2. Let A denote a symmetric positive-definite $n \times n$ matrix.

(a) Prove the following:

$$\begin{aligned} a_{ii} &> 0, \quad \text{for all } i \\ |a_{ij}| &\leq \sqrt{a_{ii}a_{jj}}, \quad \text{for all } i \text{ and } j \\ \max_{i,j} |a_{ij}| &= \max_i a_{ii}. \end{aligned}$$

(b) Show that if Gaussian elimination without interchanges is applied to A, then the remaining matrix is symmetric positive definite at every step. Hence show that there exists a unit lower-triangular L and upper triangular U such that $A = LU$.

(c) If A is factorized using Gaussian elimination without interchanges, show that the growth factor ρ_n satisfies $\rho_n \leq 1$.

Question 1.3. Assume that A is an $m \times n$ matrix with rank k ($k < \min(m, n)$).

(a) Define what is meant by a *full-rank factorization* $A = BC$.

(b) Derive a full-rank factorization of A in terms of the singular value decomposition. (You may assume that the decomposition is computed in exact arithmetic.)

(c) Using the singular-value decomposition of part (b), define bases for the subspaces range(A) and null(A). Prove that the proposed bases satisfy the properties of a subspace basis.

(d) Derive the pseudoinverse of A in terms of the full-rank factorization of part (b).

(e) Using the singular-value decomposition of part (b), define orthogonal projections onto range(A) and null(A). Prove that the proposed projections satisfy the properties of an orthogonal projection.

2. Nonlinear Equations and Optimization

Question 2.1.

(a) Derive Newton's method for finding the reciprocal of a given nonzero scalar a.

(b) Determine the exact order of convergence and asymptotic error constant for the method derived in part (a). (Do not attempt to derive the general rate-of-convergence result for Newton's method.)

Question 2.2. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(x) = x_1^2 + x_2^2 \cos x_3 - e^{x_2} x_3^2 + 4x_3.$$

(a) Compute the spectral decomposition of the Hessian matrix of second derivatives at $\bar{x} = (0, 1, 0)^T$.

(b) Compute the Newton direction p^N and modified Newton direction p^M at $\bar{x}$. Determine if p^N and p^M are descent directions.

(c) Find a direction of negative curvature that is a direction of decrease for f at $\bar{x}$.

Question 2.3. Let $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function on an open convex set $\mathcal{D}$. Let $\nabla f(x)$ denote the gradient of f at any $x \in \mathcal{D}$. If x_k is any point in $\mathcal{D}$. Consider the quadratic model

$$q_k(x) = f(x_k) + \nabla f(x_k)^T (x - x_k) + \frac{1}{2}(x - x_k)^T B(x - x_k),$$

where B is a given *fixed* symmetric positive-definite matrix.

(a) Find the vector p_k such that $x = x_k + p_k$ minimizes $q_k(x)$, and show that p_k is a descent direction for $f(x)$ at x_k.

(b) Show that p_k is a solution of the problem

$$\underset{\substack{p \in \mathbb{R}^n \\ p \neq 0}}{\text{minimize}} \quad \frac{\nabla f(x)^T p}{\|p\|_B}$$

where $\|p\|_B = (p^T B p)^{1/2}$.

(c) Given the direction p_k of part (a), formulate a back-tracking line search that will guarantee a reduction in f that is no worse than η_s times the reduction predicted by the quadratic model q_k, where η_s is a pre-assigned constant such that $0 < \eta_s < 1$. Show that the quadratic model predicts a decrease in f for all α_k such that $0 < \alpha_k < 2$.