Real Analysis, UCSD

Qualify Exam

September, 2009

Name: ____________________

Student #: ____________________

Problem	Points
Page 2 (14 points)	
Page 3 (14 points)	
Page 4 (7 points)	
Page 5 (10 points)	
Page 6 (10 points)	
Page 7 (15 points)	
Page 8 (15 points)	
Page 9 (15 points)	
Total (100 points)	

General instructions: 3 hours. Be sure to carefully motivate all (nontrivial) claims and statements. You may use without proof any result proved in the text (as well as ones covered in the lecture). If you use a theorem from the text (or lecture), refer to it either by name (if it has one) or explain what it says. Also verify explicitly all hypotheses in the theorem. You need to reprove any result given as an exercise (unless it has been singled out and lectured upon). If the statement you are asked to prove is exactly a result covered in the class you are asked to re-construct the proof instead of just citing the result as the proof.

Notations: m (or dm, dx, dy) denotes the Lebesgue measure of the Euclidean spaces. $\mathcal{D}(\mathbb{R}^n)$ denotes the smooth functions with compact support on $\mathbb{R}^n$.

(1) Determine if the statements below are **True** or **False**. If **True**, give a brief proof. If **False**, give a counterexample (or prove your assertion in another way, if you prefer). If your claim follows from a theorem in the text, name the theorem (or describe it otherwise) and explain carefully how the conclusion follows.

a) (7 pts) Let $(X, \mathcal{M}, \mu)$ be a complete measure space. If $f_n, g_n, g, f \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $|f_n| \le g_n$ and $\int g_n \, d\mu = A < \infty$ for some $A > 0$, then $\int f_n \, d\mu \to \int f \, d\mu$.

b) (7 pts) The iterated integrals

$$\int_{-1}^{1} \left[\int_{-1}^{1} \frac{xy}{(x^2+y^2)^2} dx \right] dy = \int_{-1}^{1} \left[\int_{-1}^{1} \frac{xy}{(x^2+y^2)^2} dy \right] dx.$$

Hence by the Fubini-Tonelli theorem $\frac{xy}{(x^2+y^2)^2}$ is (Lebesgue) integrable on $[-1, 1] \times [-1, 1]$.

c) (7 pts) Assume that f is continuous real valued function on $\mathbb{R}$ and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

d) (7 pts) Let X be an infinite dimensional Banach space. Then every nonempty weak*-open set in X^* is unbounded with respect to the induced norm.

e) (7 pts) A bounded sequence in a Hilbert space contains a weakly convergent subsequence.

(2) (10 pts) Assume that f_n is a sequence of measurable functions on $(X, \mathcal{M}, \mu)$. Assume that there exists a integrable function F such that $|f_n| \leq F$ μ-a.e., and $f_n \to f$ μ-a.e.. Show that $f_n \to f$ in measure.

(3) (10 pts) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on the linear vector space X. Assume that every continuous linear functional f of $(X, \|\cdot\|_1)$ is also a continuous linear function of $(X, \|\cdot\|_2)$. Prove that there exists $\alpha > 0$ such that $\|x\|_1 \leq \alpha\|x\|_2$ for all $x \in X$.

(4) (15 pts) If $f \in L^1((-\infty, -\delta) \cup (\delta, \infty))$ for every $\delta > 0$ define its *principle value integral* to be

$$PV \int_{-\infty}^{\infty} f(x)\, dx = \lim_{\delta \to 0} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) f(x)\, dx,$$

if the limit exists. For $\phi \in \mathcal{D}(\mathbb{R})$, put $\Lambda(\phi) = \int_{-\infty}^{\infty} \phi(x) \log |x|\, dx$. Show that

$$\text{(a) (8pts)} \quad \Lambda'(\phi) = PV \int_{-\infty}^{\infty} \frac{\phi(x)}{x}\, dx,$$

$$\text{(b) (7pts)} \quad \Lambda''(\phi) = -PV \int_{-\infty}^{\infty} \frac{\phi(x) - \phi(0)}{x^2}\, dx.$$

(5) (15 pts) The following provides steps to give an alternate proof of the Lebesgue-Radon-Nikodym theorem.

Suppose that μ and ν are positive finite measures on $(X, \mathcal{M})$ and let $\lambda = \mu + \nu$.

(a)(3pts) The map $f \to \int f d\nu$ is a bounded linear functional on $L^2(\lambda)$ so there exists $g \in L^2(\lambda)$ such that for any $f \in L^2(\lambda)$, $\int f(1-g)\, d\nu = \int fg\, d\mu$;

(b)(5 pts) $0 \le g \le 1$ λ-a.e., so we can assume that $0 \le g \le 1$ everywhere;

(c)(4 pts) Let $A = \{x : g < 1\}$, $B\{x : g(x) = 1\}$, and set $\nu_a(E) = \nu(A \cap E)$, $\nu_s(E) = \nu(B \cap E)$. Then $\nu_s \perp \mu$ and $\nu_a << \mu$;

d)(3 pts) Moreover $d\nu_a = g(1-g)^{-1}\chi_A d\mu$.

(6) (15 pts) Suppose that on $\mathbb{R}^n$, $|\phi(x)| \leq C(1+|x|)^{-n-\epsilon}$ for some positive C and $\epsilon > 0$. Also assume that $\phi(x)$ is measurable. If $f \in L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$, define

$$M_\phi(f)(x) \doteq \sup_{t>0} |f * \phi_t(x)|$$

where $f * \phi_t(x) = \int f(x-y)\phi_t(y)\,dy$, $\phi_t(y) = \frac{1}{t^n}\phi(\frac{y}{t})$. Show that there exists C', independent of f such that $M_\phi(f) \leq C'H(f)$ where $H(f)$ is the Hardy-Littlewood maximum function defined as

$$H(f)(x) \doteq \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\,dy.$$

END OF EXAM