160 points Complex Analysis Qualifying Examination Sept. 11, 2009

JUSTIFY EVERYTHING BY CITATION OR PROOF.

(20) 1. Find the principal value of $\log(e^{3+7i})$.

(20) 2. Suppose that $f(z)$ is defined and analytic in the punctured neighborhood of zero, $N = \{z \mid 0 < |z| < 1\}$ and that for all z in N, $|f(z)| < 1$. Provide a proof of the standard result that $f(z)$ has a removable singuarity at $z = 0$.

(20) 3. Evaluate $\prod_{n=2}^{\infty}\left(1 - \frac{1}{n^2}\right)$ in two different ways.

(20) 4. Suppose $\{f_n(z)\}_{n\geq 1}$ is a sequence of analytic functions on a reqion A which converges uniformly on A to a function $f(z)$. Show that $f(z)$ is analytic on A and that the sequence of derivatives $\{f_n'(z)\}_{n\geq 1}$ converges uniformly to $f'(z)$ on compact subsets of A.

(20) 5. Evaluate the integral $\int_0^{\infty} \frac{x\,dx}{x^4 + 1}$ via residue theory.

It is not necessary to simplify your answer.

(20) 6. Suppose that $f(z) = u(z) + iv(z)$ is an entire function with real part $u(z)$ and imaginary part $v(z)$ such that for all z,

$$u(z) + v(z) < 1.$$

Prove that $f(z)$ is a constant.

(20) 7. Suppose ρ is a third root of 1 (other than 1). Let

$$f(z) = \sin(z) + \sin(\rho z) + \sin(\rho^2 z).$$

Prove that $f(z)$ has a zero other than $z = 0$.

(20) 8. We are given two sequences of complex numbers, $\{\alpha_j\}_{j\geq 1}$ and $\{\beta_j\}_{j\geq 1}$, where the α_j are all distinct and $|\alpha_j| \to \infty$ as $j \to \infty$. Use a combination of the Mittag-Leffler Theorem and the Weierstrass Product Theorem, or any other method, to show that there exists an entire function $f(z)$ such that for each j, $f(z)$ takes the value β_j at α_j with multiplicity at least 2.

(5 bonus points) if you can get $f(z)$ which for all j takes the value β_j at $z = \alpha_j$ WITH MULTIPLICITY EXACTLY 2. ($f(z)$ takes the value β at $z = \alpha$ with multiplicity exactly 2 if $f(z) - \beta$ has a zero at $z = \alpha$ of order precisely 2.)