Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiners: Philip E. Gill and Bo Li

9am–Noon
Wednesday September 5, 2007
5402 AP&M

NAME ____________________

#1.1	30	
#1.2	30	
#1.3	30	
#2.1	30	
#2.2	30	
#2.3	30	
#3.1	30	
#3.2	30	
Total	240	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition Numbers and Linear Equations

Question 1.1.

(a) Assume that $A \in \mathbb{C}^{m\times n}$. Define the one-norm $\|A\|_1$, two-norm $\|A\|_2$, and infinity norm $\|A\|_\infty$ of A. Show that

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}|.$$

(b) Assume that $D \in \mathbb{C}^{n\times n}$ with $D = \text{diag}(d_1, d_2, \ldots, d_n)$. Prove that the matrix p-norm is such that $\|D\|_p = \max_{1\le i\le n} |d_i|$ for all $1 \le p \le \infty$.

Question 1.2.

(a) Consider the subtraction $x = a - b$ of two real numbers a and b such that $a \ne b$. Suppose that $\tilde{a}$ and $\tilde{b}$ are the result of making a *relative* perturbation Δa and Δb to a and b. Find the relative error of $\tilde{x} = \tilde{a} - \tilde{b}$ as an approximation to x and hence find a condition number for the operation of subtraction. *Assume that all calculations are done in exact arithmetic.*

(b) State the *standard rounding-error model* for floating-point arithmetic. Given three representable numbers a, b and c, compute the backward and forward relative error for the floating-point value $\hat{s}$ of the expression $s = ab+c$. Describe a situation in which $\hat{s}$ has large forward error, but small backward error.

Question 1.3. Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite.

(a) Prove the following:

(i) $a_{ii} > 0$, for all i;

(ii) $|a_{ij}| \le \sqrt{a_{ii}a_{jj}}$, for all i and j; and

(iii) $\max_{i,j} |a_{ij}| = \max_i a_{ii}$.

(b) Prove that A may be factorized as $A = LDL^T$, where L is unit lower-triangular and D is diagonal with positive diagonal elements.

2. Nonlinear Equations and Optimization

Question 2.1.

(a) Define the Frobenius norm of $A \in \mathbb{C}^{m\times n}$.

(b) Prove that for any $s \in \mathbb{R}^n$, it holds that $\|ss^T\|_F = \|s\|_2^2$. Hence show that $U^* = (y - As)s^T/s^Ts$ solves the optimization problem

$$\min \{ \|U\|_F \mid U \in \mathbb{R}^{n\times n},\ (A+U)s = y \},$$

where $A \in \mathbb{R}^{n\times n}$, and s and y are given fixed vectors in $\mathbb{R}^n$.

(c) Give a brief discussion of the significance of part (b) in reference to Broyden's method for multivariate zero finding.

Question 2.2. Let $f : D \subseteq \mathbb{R}^n \mapsto \mathbb{R}$ be a twice continuously differentiable function with gradient vector $\nabla f(x)$.

(a) Consider the quadratic model $q_k(x) = b_k + a_k^T(x - x_k) + \frac{1}{2}(x - x_k)^T B(x - x_k)$, where b_k is a scalar, a_k is an n-vector and B is a given *fixed* symmetric positive-definite matrix. Find the values of a_k and b_k that define a model with function value and gradient equal to $f(x_k)$ and $\nabla f(x_k)$. Find the solution p_k of the quadratic subproblem

$$\underset{p\in\mathbb{R}^n}{\text{minimize}}\ \ q_k(x_k + p).$$

Prove that p_k is a descent direction for $f(x)$ at x_k.

(b) Prove that p_k is a solution of the problem

$$\underset{\substack{p\in\mathbb{R}^n\\ p\neq 0}}{\text{minimize}}\ \ \frac{p^T\nabla f(x_k)}{\|p\|_B},$$

where $\|p\|_B = (p^TBp)^{1/2}$. Briefly discuss the significance of this result.

Question 2.3. Let $F : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}^m$ be a continuously differentiable function on an open convex set $\mathcal{D}$. We seek a zero of F by minimizing the scalar-valued function $f(x) = \|F(x)\|_2$. Let x_k and p_k denote vectors in $\mathbb{R}^n$ such that $x_k \in \mathcal{D}$ and $p_k \neq 0$.

(a) If $F(x_k) \neq 0$ and φ is the univariate function $\varphi(\alpha) = \|F(x_k + \alpha p_k)\|_2$, find an expression for the directional derivative $\varphi'(\alpha)$ in terms of $F(x_k + \alpha p_k)$ and $F'(x_k + \alpha p_k)$.

(b) If p_k is the least-length solution of $\min_p \|F(x_k) + F'(x_k)p\|_2$, derive the conditions under which p_k is a descent direction for $\|F\|_2$ at x_k.

(c) Derive the termination criterion for a backtracking line search to be used in conjunction with the direction p_k defined in part (b). Derive the backtracking termination criterion for the special case where $F : \mathcal{D} \subseteq \mathbb{R}^n \mapsto \mathbb{R}^n$ and $\text{rank}(F') = n$.

3. Approximation and Numerical ODEs

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We also denote by $\mathcal{P}_n$ the set of all polynomials of degree $\le n$ for an integer $n \ge 0$.

Question 3.1.

(a) Let $f \in C^1[a,b]$ and $\varepsilon > 0$. Prove that there exists a polynomial p such that

$$\max_{a \le x \le b} |f(x) - p(x)| < \varepsilon \qquad \text{and} \qquad \max_{a \le x \le b} |f'(x) - p'(x)| < \varepsilon.$$

(b) Find the least-squares approximation in $\mathcal{P}_1$ of the function $f(x) = x^4$ in $L^2[-1,1]$.

Question 3.2.

(a) Let $k \ge 1$ be an integer. Suppose $p_k, q_k \in \mathcal{P}_k$ are the Lagrange interpolation polynomials that interpolate $f_0, \dots, f_k$ at $x_0, \dots, x_k$ and $f_1, \dots, f_{k+1}$ at $x_1, \dots, x_{k+1}$, respectively. Define

$$r_{k+1}(x) = \frac{(x - x_0)q_k(x) - (x - x_{k+1})p_k(x)}{x_{k+1} - x_0}.$$

Prove that $r_{k+1}(x)$ is the Lagrange interpolation polynomial that interpolates $f_0, \dots, f_k$, and f_{k+1} at $x_0, \dots, x_k$, and x_{k+1}.

(b) The trapezoidal numerical integration rule is given by

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\left[f(a) + f(b)\right].$$

Let $f \in C^2[a,b]$.

Prove that there exists $\xi \in (a,b)$ such that

$$\int_a^b f(x)\,dx = \frac{1}{2}(b-a)\left[f(a) + f(b)\right] - \frac{1}{12}(b-a)^3 f''(\xi).$$

Let $N \ge 1$ be an integer, $h = (b-a)/N$, and $x_k = a + kh$, $k = 0, \dots, N$. Prove that there exists $\eta \in (a,b)$ such that

$$\int_a^b f(x)\,dx = \left\{ \frac{h}{2}\left[f(a) + f(b)\right] + h \sum_{k=1}^{N-1} f(x_k) \right\} - \frac{(b-a)f''(\eta)}{12} h^2.$$