Complex Analysis Qualifying Exam September, 2006

Name:	____________________
Student #:	____________________
TA's Name:	____________________
Session #:	____________________

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General guidelines: You may cite without proof any theorem given in the text (unless explicitly stated otherwise). You need to reprove any result given in an exercise. If in doubt, please ask!

1. (15 points) If $f(z)$ is analytic in $|z| < 1$, continuous on $|z| \leq 1$ and satisfies $|f| = 1$ on $|z| = 1$, show that $f(z)$ is rational.

   Hint: First consider the case that $f(z)$ has no zero.

2. (20 points) Assume the following representation formula

$$u(z) = \operatorname{Re}\left[\frac{1}{2\pi i}\int_{|\eta|=R}\frac{\eta - z}{\eta + z}u(\eta)\frac{d\eta}{\eta}\right]$$

for any harmonic function u defined on $|z| < R$ and continuous up to $|z| \leq R$.

a) If $u(z)$ is a harmonic function defined on $\mathbb{C}$ such that

$$\lim_{z\to\infty}\frac{|u(z)|}{|z|} = 0$$

show that u must be a constant.

b) Show that

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2 - |a|^2}{|Re^{i\theta} - a|^2}u(Re^{i\theta})\,d\theta.$$

Hint: The solution of b) does not require a).

3. (15 points) Let Y be a simply-connected region contained inside the disk $D = \{z \,||z| < 1\}$. Assume $o \in Y \neq D$. Without appealing the Riemann mapping theorem itself prove that there exists a r with $1 > r > 0$ and a univalent conformal map $f : Y \to D(r) := \{z \,||z| < r\}$ with $f(0) = 0$ and $f'(0) = 1$.

Hint: You may use part of the proof of the Riemann mapping theorem.

4. (15 points) Assume that f is an entire function. Let $M(r) = \sup_{|z|=r} |f(z)|$. Assume that the discrete sequence $\{a_n\}$ are zeros of f and $f(0) = 1$. Show that

$$k \log 2 \leq \log\left(M(2|a_k|)\right)$$

5. (15 points) How many roots of the equation $z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half plane? Justify your answer.

6. (10 points) Let $f(z)$ be a holomorphic function in the disk $|z| < 2$. Show that $\sum_1^\infty \frac{f^{(n)}(z)}{n!}$ converges uniformly in any compact subset of the unit disk $|z| < 1$.

7. (10 points) Suppose that $a_n \to \infty$ and A_n are arbitrary complex numbers. Show that there exists an entire function $f(z)$ which satisfies $f(a_n) = A_n$.

END OF EXAM