

Department of Mathematics
MA/PhD Qualifying Examination in Applied Algebra
Part I

Examiners: Philip Gill and Cristian Popescu

1:00–2:15pm, AP&M 7421

Tuesday September 5, 2006

NAME _____

#1.1	20	
#1.2	20	
#2.1	20	
#2.2	20	
Total	80	

- Do all problems.
- Add your name in the space provided and staple this page to your solutions for Section #1.
- Start your solutions for the questions in Section #2 on a fresh page.
- Write your name clearly on every sheet submitted.

1. Linear Algebra

Question 1.1. Assume that (λ, x) is an eigenpair of $A \in M_n$ such that $\operatorname{am}(\lambda) = \operatorname{gm}(\lambda) = 1$. Prove that there exists a nonsingular matrix $\begin{pmatrix} x & X \end{pmatrix}$ with inverse $\begin{pmatrix} y & Y \end{pmatrix}^*$ such that

$$\begin{pmatrix} y^* \\ Y^* \end{pmatrix} A \begin{pmatrix} x & X \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & M \end{pmatrix}.$$

Question 1.2. Given $A \in M_{m,n}$ with $m \geq n$, prove that there exists a unique $U \in M_{m,n}$ with orthonormal columns, and a unique Hermitian positive semidefinite $H \in M_n$ such that $A = UH$.

(State in detail any auxiliary results that you use.)

2. Group Theory

Question 2.1. Let G be a group and let $\mathcal{Z}(G)$ denote its center.

- Show that if $G/\mathcal{Z}(G)$ is cyclic then $G = \mathcal{Z}(G)$.
- Show that if $\operatorname{card}(G) = p^3$, for some prime number p and G is non-commutative then $\operatorname{card}(\mathcal{Z}(G)) = p$.
- Construct a non-commutative group G of cardinality (order) 16 whose center $\mathcal{Z}(G)$ is not cyclic.

Note. As usual, $\operatorname{card}(X)$ denotes the cardinality of the set X .

Question 2.2. Let n be an integer. Let G_n be the group given by generators and relations as follows.

$$G_n = \langle x, y \mid x^3 = 1, xyx^{-1} = y^n \rangle$$

- Provide (with proof) necessary and sufficient conditions on the integer n for the group G_n to be **finite**.
- Assuming that G_n is finite, compute its order as a function of n .