MATH 220. Complex Analysis
Qualifying exam. September 8, 2005

**General instructions**: 3 hours. No books or notes. Be sure to motivate all (non-trivial) claims and statements. You may use without proof any result proved in the text unless otherwise stated. You need to reprove any result given as an exercise. The notation $B(a, r) := \{z \colon |z - a| < r\}$ and $\mathbb{D} := B(0, 1)$ will be used. If $G \subset \mathbb{C}$ is an open set, $H(G)$ denotes the set of all analytic functions in $G$.

1. Find all functions $f \in H(B(0; 2))$ such that the following two conditions are satisfied:
(a) $|f(z)| = 1$ if $|z| = 1$
(b) $f$ has a zero of multiplicity 2 at $z = 1/2$ and no other zeroes.
Hint: Consider first the case where $f$ satisfies (a), but has no zeroes.

2. Let $f$ be a nonconstant analytic function in $\mathbb{D}$ with $f(0) = 0$. Show that there exist a real number $r$, $0 < r \leq 1$, a function $g \in H(B(0, r))$ with $g(0) \neq 0$, and a positive integer $m$, such that

$$f(z) = \big(zg(z)\big)^m, \qquad z \in B(0, r).$$

3. Prove that if $f$ is a non-constant analytic function on a bounded open set $G \subset \mathbb{C}$ and is continuous on $\overline{G}$, then either $f$ has a zero in $G$ or $|f(z)|$ reaches its minimum value on $\partial G$.

4. Let $K \subset G \subset \mathbb{C}$ with $K$ compact and $G$ open. Suppose that for any $f$ analytic in an open neighborhood of $K$ and any $\epsilon > 0$ there is $g \in H(G)$ so that $|f(z) - g(z)| < \epsilon$ for all $z \in K$. Let $z_0 \in G \setminus K$ be arbitrary. Show that there exists $h \in H(G)$ such that

$$|h(z_0)| > \sup_{w \in K} |h(w)|.$$

5. Use the method of residues to compute the integral $\displaystyle\int_0^\infty \frac{\sin x}{x} dx$. Justify all your steps.

Hint: Integrate the function $\dfrac{e^{iz}}{z}$ on an appropriate closed curve.

6. State and prove Harnack's inequality for non-negative functions that are continuous in $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$. (You may use without proof the Poisson integral formula.)