2004 Fall Topology Qual

Two and a half hour exam. Each question is worth five marks. You can use any standard facts or theorems in your work provided you state them clearly.

1. Find a space X that has the same integral homology and fundamental group as the torus $S^1 \times S^1$, but is not homotopy equivalent to the torus. Prove that X is not homotopy equivalent to the torus.

2. Consider the standard covering projection $S^n \to \mathbb{R}P^n$ which maps antipodal points to the same point in $\mathbb{R}P^n$. Prove that the covering projection is not null homotopic.

3. Show that $\mathbb{R}P^k$ is not a retract of $\mathbb{R}P^n$ for $k < n$.

4. Let M be a compact connected nonorientable 3-manifold. Show the first integral homology group of M is infinite.

5. Prove the Borsuk-Ulam theorem that if $n > m \geq 1$, then there is no map $g : S^n \to S^m$ which commutes with the antipodal map.

6. Let M^4 be a closed connected simply-connected 4-manifold. Show that $H_1(M;\mathbb{Z}) = H_3(M;\mathbb{Z}) = 0$ and that $H_2(M;\mathbb{Z})$ is a free abelian group.

7. Describe the universal cover of $X = \mathbb{R}P^3 \vee S^2$, and use it to compute the abelian group $\pi_2(X)$.

8. Let X be the space obtained by identifying the edges of a solid hexagon as shown below. Compute $H_*(X;\mathbb{Z})$.