Ph.D./Masters Qualifying Examination in Numerical Analysis

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1:00–4:00pm
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7421 AP&M

NAME ____________________

#1.1	20	
#1.2	20	
#1.3	20	
#2.1	20	
#2.2	20	
#2.3	20	
#3.1	20	
#3.2	20	
Total	160	

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.

1. Norms, Condition numbers and Linear Equations

Question 1.1. Assume that $A \in \mathbb{C}^{m\times n}$.

(a) Define the one-norm $\|A\|_1$ and infinity norm $\|A\|_\infty$ of A. Show that

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|.$$

(b) Establish the following identities between the one-norm and infinity-norm:

(i) $\|A\|_1 \le m\,\|A\|_\infty$.

(ii) $\dfrac{1}{n}\,\|A\|_\infty \le \|A\|_1$.

Question 1.2.

(a) State the *standard rounding-error model* for floating-point arithmetic.

(b) Let $\mathbf{u}$ denote the unit roundoff, and assume that $n\mathbf{u} < 1$ for the positive integer n. If $\{\delta_i\}$ are n scalars such that $|\delta_i| \le \mathbf{u}$, prove that

$$\prod_{i=1}^{n}(1+\delta_i) = 1+\theta_n, \quad \text{where} \quad |\theta_n| \le \gamma_n,$$

with $\gamma_n = n\mathbf{u}/(1-n\mathbf{u})$.

(c) Let $\{x_i\}$ denote any set of n representable real numbers. Perform a forward and backward rounding-error analysis for the floating-point computation of $\sum_{i=1}^{n}|x_i|$. Comment on the forward and backward stability of this calculation.

Question 1.3. Suppose that Gaussian elimination without interchanges succeeds on a symmetric matrix A. Prove that the remaining matrix must be symmetric at each step. Hence show that if A is symmetric positive definite, then Gaussian elimination without interchanges gives $\rho_n \le 1$.

2. Least-Squares and Eigenvalues

Question 2.1. Let A be any nonzero $m \times n$ matrix of rank r.

(a) State the definition of a full-rank factorization of A.

(b) Suppose that A has a full-rank factorization $A = FG$, where F is $m \times r$ and G is $r \times n$. Show that $A^\dagger = G^\dagger F^\dagger$.

(c) Use the result of part (a) to derive the least-length least-squares solution of $Ax \approx b$ using the singular-value decomposition.

Question 2.2. Consider a non-defective matrix $A \in \mathbb{C}^{2\times 2}$ such that

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}.$$

(a) Find the left and right eigenvectors of A.

(b) Find the condition number of each of the eigenvalues of A.

Question 2.3.

(a) Given any nonzero vector $u \in \mathbb{C}^n$, show that the matrix $I - uu^H/\beta$ with $\beta = \frac{1}{2}u^H u$ is Hermitian and unitary.

(b) Use part (a) to show that for any vector $x \in \mathbb{C}^n$, there exists a unitary matrix H such that $Hx = \gamma e_1$, where e_1 is the first column of the identity and $|\gamma| = \|x\|_2$. Hence show that if $\|x\|_2 = 1$, then there exists a unitary matrix with x as its first column.

(c) Let x denote an approximate eigenvector of A with $\|x\|_2 = 1$. If Q is unitary with x as its first column, show that if the product $Q^H AQ$ is partitioned as

$$Q^H AQ = \begin{pmatrix} x^H Ax & b^H \\ e & C \end{pmatrix},$$

then $\|e\|_2 = \|Ax - (x^H Ax)x\|_2 \le \|Ax - \sigma x\|_2$ for all σ. Briefly discuss the relevance of this inequality to the accuracy of the Rayleigh quotient as an estimate of an eigenvalue associated with an approximate eigenpair (λ, x).

3. Interpolation, Approximation and ODEs

Question 3.1. Consider the quadrature formula

$$\int_{-1}^{1} f(x)\,dx \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right).$$

(a) Verify that it is exact for all polynomials of degree 5 or less.

(b) Is it a Gaussian quadrature? Why or why not?

Question 3.2. Let $\{\phi_k, k = 0, 1, \ldots, \infty\}$ be an orthogonal set of polynomials in the interval $[a, b]$, where ϕ_k has degree k.

(a) Given $n \geq 0$, show $\{\phi_0, \phi_1, \ldots, \phi_n\}$ forms a basis for the set of polynomials P_n of degree less than or equal to n in $[a, b]$.

(b) Given $n \geq 1$, show $\phi_{n+1} = (A_{n+1}x + B_{n+1})\phi_n + C_{n+1}\phi_{n-1}$ holds in $[a, b]$, for some constants A_{n+1}, B_{n+1}, and C_{n+1}. Hint: Consider subtracting ϕ_{n+1} by a certain constant multiple of $x\phi_n$ and using part (a).