2002 Fall Topology Qual

1. Let $E = \mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R}$ be the subset of the plane whose points have at least one of the coordinates an integer. Let $S^1 \vee S^1 \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ be the one-point union of circles. Define $p : E \to S^1 \vee S^1$ by

$$p(x, y) = (e^{2\pi i x}, e^{2\pi i y})$$

(a). Verify that p is a covering space map.

(b). Let $\sigma : (I, 0) \to (E, (0, 0))$ be the loop which traverses the unit square $I \times \{0, 1\} \cup \{0, 1\} \times I$ once counterclockwise. Prove that $p\sigma$ is the commutator of the two loops of the figure-eight.

(c). Prove that $p_{\#} : \pi_1(E, (0, 0)) \to \pi_1(S^1 \vee S^1, (1, 1))$ is a monomorphism. Show that this implies that $\pi_1(S^1 \vee S^1, (1, 1))$ is not abelian.

2. Let T be the torus which is obtained by identifying the edges of the unit square in the usual manner. Let $S^1 \vee S^1$ be the one-point union of circles which is the image of the boundary of the unit square.

(a). Compute $H_*(T, S^1 \vee S^1; \mathbb{Z})$ and the map $i_* : H_*(S^1 \vee S^1; \mathbb{Z}) \to H_*(T; \mathbb{Z})$, where $i : S^1 \vee S^1 \hookrightarrow T$ is the inclusion map.

(b). Let $Z = S^1 \vee S^1 \vee S^2$ be the one-point union of the two circles and a 2-sphere. Prove that $H_*(Z; \mathbb{Z})$ and $H_*(T; \mathbb{Z})$ are isomorphic, but that Z and T do not have the same homotopy type.

3. (a). Construct a space Y with the following properties:

$$H_k(Y; \mathbb{Z}) = \begin{cases} \mathbb{Z}_4 & \text{if } k = 2 \\ \mathbb{Z} & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(b). Compute $H_*(\mathbb{R}P^2 \times Y; \mathbb{Z}_2)$, $H_*(\mathbb{R}P^2 \times Y; \mathbb{Z})$, and $H^*(\mathbb{R}P^2 \times Y; \mathbb{Z})$.

4. Let X be a finite-dimensional cell complex with only even-dimensional cells. Prove that $H_*(X; \mathbb{Z})$ is torsion-free.

5. Prove that any continuous map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point, if $n \geq 1$.

6. Let X be an n-dimensional $\mathbb{Z}_3$-orientable manifold. Prove that X is orientable.

7. Describe submanifold representatives of the generators of the homology groups of $\mathbb{C}P^n$, and explain how to use these to determine the cohomology ring structure.

8. Suppose K is a knot (a smoothly-embedded image of the circle S^1) in S^4. Use transversality to compute the fundamental group of the complement $S^4 - K$.

9. Let M be an n-dimensional compact connected manifold. Suppose all cup products vanish in $H^*(M;\mathbb{Z})$. Prove that $H^*(M;\mathbb{Z})$ is isomorphic to $H^*(S^n;\mathbb{Z})$.

10. Use transversality to prove that there is no smooth retraction $r : B^n \to S^{n-1}$, and consequently (the Brouwer fixed point theorem) that any smooth automorphism of B^n has a fixed point.