Numerical Analysis Qualifying Exam
Part AB
September 10, 2002

Print Name ____________________

Signature ____________________

# 1	25	
# 2	20	
# 3	25	
# 4	10	
# 5	10	
# 6	15	
Part AB	105	
Part C	45	
Total	150	

1. (15) (a) State and prove the Schur Decomposition Theorem.

(10) (b) Use it to prove: A has n orthonormal eigenvectors iff $A^H A = AA^H$, where $A \in \mathbb{C}^{n \times n}$.

(20) 2. (a) Let A be $m \times n, m > n, B = [A|z]$. Show that $\sigma_1(B) \geq \sigma_1(A)$ and $\sigma_{n+1}(B) \leq \sigma_n(A)$.

(b) Let A be $m \times n, m \geq n, C = \begin{bmatrix} A \\ v^T \end{bmatrix}$. Show that $\sigma_n(C) \geq \sigma_n(A)$ and $\sigma_1(A) \leq \sigma_1(C) \leq \sqrt{\sigma_1(A)^2 + v^T v}$.

(10) 3. (a) Use Gershgorin's Theorem to prove that a real symmetric diagonally dominant matrix with positive diagonal elements is positive definite.

(15) (b) Show that if the single shift QR method converges, then the convergence is:
(a) quadratic for general matrices
(b) cubic for symmetric matrices

(10) 4. Prove that $\|B(\lambda) - A^+\|_2 = \frac{\lambda}{\sigma_r(\sigma_r^2+\lambda)}$, where $B(\lambda) = (A^T A + \lambda I)^{-1} A^T, \lambda > 0, A$ is $m \times n, m \geq n, r = \text{rank}(A)$.

(10) 5. Let A be $n \times n$, nonsingular, and $A = QR$, where Q is orthogonal and R is upper triangular with positive diagonal. Prove that Q and R are unique.

(15) 6. Prove that if A is symmetric positive definite, $\max_{i,j} |a_{ij}| = 1$, then $\max_{i,j,k} |a_{ij}^{(k)}| = 1$ under LDL^T (or LU) decomposition.

Part C

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NAME ______________

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#3	15	
Total	45	

Question 1. Let $f \in C^4(a,b)$, and let $x_0 = a < x_1 < \ldots < x_{n-1} < x_n = b$. Let s be the C^2 natural cubic spline interpolant of f and let g be any other C^2 function satisfying $g(x_i) = f(x_i)$, $0 \le i \le n$, $g''(x_0) = g''(x_n) = 0$. Prove

$$\| s'' \|_{\mathcal{L}^2(a,b)} \le \| g'' \|_{\mathcal{L}^2(a,b)} .$$

Question 2. Find the one-point Gauss-Quadrature Rule of the form

$$\int_0^1 f(x)\sqrt{x}\,dx \approx A\,f(\alpha).$$

Question 3. Define the terms:

a. Consistency

b. Stability

c. Convergence

as they relate to a multi-step formula for solving the initial value problem $y' = f(y)$, $y(0) = y_0$. Apply these concepts to analyze the two step formula

$$y_{k+1} = y_{k-1} + 2hf(y_k).$$