# Applied Algebra Qualifying Exam
# Spring 2015

**Instructions:** Do all of Problems 1-4. Choose two among Problems 5-7 to do. (If you attempt all three of Problems 5-7, only the best two will count towards your score on this exam.) All problems are weighted equally. Good luck!

**Problem 1:** Let $D_5 = \langle r, s : s^2 = r^5 = 1, srs = r^{-1} \rangle$ denote the group of symmetries of a regular pentagon. Find the character table of $D_5$.

**Problem 2:**

(1) Write down the character table of the product of symmetric groups $\mathfrak{S}_3 \times \mathfrak{S}_2$.

(2) Let $S^{(2,2,1)}$ be the irreducible representation of $\mathfrak{S}_5$ indexed by the partition $(2, 2, 1) \vdash 5$. Determine the decomposition of the restriction $S^{(2,2,1)} \downarrow_{\mathfrak{S}_3 \times \mathfrak{S}_2}$ into irreducible $\mathfrak{S}_3 \times \mathfrak{S}_2$-representations.

**Problem 3:**

(1) Write down the character table of the symmetric group $\mathfrak{S}_4$.

(2) For $\lambda \vdash 4$, let $S^\lambda$ denote the corresponding irreducible representation of $\mathfrak{S}_4$. Endow the tensor product $S^{(3,1)} \otimes S^{(2,2)} \otimes S^{(2,1,1)}$ with a $\mathfrak{S}_4$-module structure by

$$\sigma.(u \otimes v \otimes w) := (\sigma.u) \otimes (\sigma.v) \otimes (\sigma.w)$$

for $\sigma \in \mathfrak{S}_4, u \in S^{(3,1)}, v \in S^{(2,2)}$, and $w \in S^{(2,1,1)}$. (This is the *Kronecker product*.) Find the decomposition of $S^{(3,1)} \otimes S^{(2,2)} \otimes S^{(2,1,1)}$ into irreducible $\mathfrak{S}_4$-modules.

(3) Describe the structure (as a product of matrix algebras over $\mathbb{C}$) of the algebra of $\mathfrak{S}_4$-endomorphisms $\text{End}_{\mathfrak{S}_4}(S^{(3,1)} \otimes S^{(2,2)} \otimes S^{(2,1,1)})$.

**Problem 4:** Let $G$ be a finite group and let $X : G \to GL_n(\mathbb{C})$ be a complex matrix representation of $G$. For any $g \in G$, prove that the matrix $X(g)$ is diagonalizable. Is this still true if the group $G$ is infinite?

**Problem 5:** Let $\mathbb{K}$ be an algebraically closed field, let $\mathbb{K}^n$ be affine $n$-space over $\mathbb{K}$, and consider the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$.

(1) Suppose that $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ is an ideal such that $\mathbf{V}(I) \subseteq \mathbb{K}^n$ is a finite set. Prove that $\mathbb{K}[x_1, \ldots, x_n]/I$ is a finite-dimensional $\mathbb{K}$-vector space.

(2) Let $\mathbb{F}$ be a field which is *not* algebraically closed. Prove that there exists an ideal $I \subseteq \mathbb{F}[x, y]$ such that $\mathbf{V}(I) = \emptyset$ but $\mathbb{F}[x, y]/I$ is an infinite-dimensional $\mathbb{F}$-vector space.

**Problem 6:** Consider the matrix $A \in GL_3(\mathbb{C})$ given by

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $G$ be the order 3 cyclic group generated by $A$.

(1) Use Molien's Theorem to find the Hilbert series of the invariant ring $\mathbb{C}[x, y, z]^G$.

(2) Find two finite subgroups $H, K \leq GL_3(\mathbb{C})$ such that $H$ and $K$ are isomorphic as abstract groups but the invariant rings $\mathbb{C}[x,y,z]^H$ and $\mathbb{C}[x,y,z]^K$ are *not* isomorphic as graded algebras. (Hint: There is an example where $|H| = |K| = 2$.)

**Problem 7:** Let $I \subseteq \mathbb{C}[x,y]$ be the ideal given by $I = \langle y^2 + xy, xy^2 + x^2y + x^2 \rangle$.

(1) Find the reduced Gröbner basis for $I$ with respect to the lexicographic order where $y > x$.
(2) Find the reduced Gröbner basis basis for the ideal $I \cap \mathbb{C}[x]$.
(3) Find a $\mathbb{C}$-linear basis for the $\mathbb{C}$-vector space $\mathbb{C}[x,y]/I$.